

HOMOGENIZED SPECTRAL PROBLEMS FOR EXACTLY SOLVABLE OPERATORS: ASYMPTOTICS OF POLYNOMIAL EIGENFUNCTIONS

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ABSTRACT. Consider a homogenized spectral pencil of exactly solvable linear differential operators $T_\lambda = \sum_{i=0}^k Q_i(z) \lambda^{k-i} \frac{d^i}{dz^i}$, where each $Q_i(z)$ is a polynomial of degree at most i and λ is the spectral parameter. We show that under mild nondegeneracy assumptions for all sufficiently large positive integers n there exist exactly k distinct values $\lambda_{n,j}$, $1 \leq j \leq k$, of the spectral parameter λ such that the operator T_λ has a polynomial eigenfunction $p_{n,j}(z)$ of degree n . These eigenfunctions split into k different families according to the asymptotic behavior of their eigenvalues. We conjecture and prove sequential versions of three fundamental properties: the limits $\Psi_j(z) = \lim_{n \rightarrow \infty} \frac{p'_{n,j}(z)}{\lambda_{n,j} p_{n,j}(z)}$ exist, are analytic and satisfy the algebraic equation $\sum_{i=0}^k Q_i(z) \Psi_j^i(z) = 0$ almost everywhere in \mathbb{CP}^1 . As a consequence we obtain a class of algebraic functions possessing a branch near $\infty \in \mathbb{CP}^1$ which is representable as the Cauchy transform of a compactly supported probability measure.

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1. INTRODUCTION AND MAIN RESULTS

In this paper we study the properties of asymptotic root-counting measures for families of (nonstandard) polynomial eigenfunctions of exactly solvable linear differential operators. Using this information we describe a class of algebraic functions possessing a branch near $\infty \in \mathbb{CP}^1$ which is representable (up to a constant factor) as the Cauchy transform of a compactly supported probability measure.

Notation 1. A linear ordinary differential operator $T := \sum_{i=1}^k a_i(z) \frac{d^i}{dz^i}$ is called *exactly solvable* or *ES* for short if T (nonstrictly) preserves the infinite flag $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \dots$, where \mathcal{P}_i denotes the linear space of polynomials in z of degree at most i , see [27].

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The well-known classification theorem of S. Bochner, see [9] and [21], states that each coefficient $a_i(z)$ of an exactly solvable T is a polynomial of degree at most i .

T is called *strictly exactly solvable* if the flag $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \dots$ is strictly preserved. One can show that the coefficients $a_i(z) := \sum_{j=0}^i a_{i,j} z^j$ of a strictly exactly solvable T satisfy the condition that the equation $\sum a_{i,i} t(t-1) \dots (t-i+1) = 0$ has no positive integer solutions, see Lemma 1 below.

Finally, let ES_k denote the linear space of all ES -operators of order at most k .

Consider a spectral pencil of ES_k -operators $T_\lambda := \sum_{i=0}^k a_i(z, \lambda) \frac{d^i}{dz^i}$, i.e., a parametrized curve $T : \mathbb{C} \rightarrow ES_k$. Each value of the parameter λ for which the equation $T_\lambda p(z) = 0$ has a polynomial solution is called a (*generalized*) *eigenvalue* and the corresponding polynomial solution is called a (*generalized*) *eigenpolynomial*.

The main problem that we address in this paper is as follows.

Problem 1. *Given a spectral pencil T_λ describe the asymptotics of its eigenvalues and the asymptotics of the root distribution of the corresponding eigenpolynomials.*

Motivated by the necessities of the asymptotic theory of linear ordinary differential equations, see e.g. [15, Ch. 5], we concentrate below on the fundamental special case of *homogenized spectral pencils*, i.e., rational normal curves in ES_k of the form

$$T_\lambda = \sum_{i=0}^k Q_i(z) \lambda^{k-i} \frac{d^i}{dz^i}, \quad (1.1)$$

where each $Q_i(z)$ is a polynomial of degree at most i . Consider the algebraic curve Γ given by the equation

$$\sum_{i=0}^k Q_i(z) w^i = 0, \quad (1.2)$$

where the polynomials $Q_i(z) = \sum_{j=0}^i a_{i,j} z^j$ are the same as in (1.1). Given a curve Γ as in (1.2) and the pencil T_λ as in (1.1) with the same coefficients $Q_i(z)$, $0 \leq i \leq k$, we call Γ the *plane curve associated with T_λ* and we say that T_λ is the *spectral pencil associated with Γ* .

The curve Γ and its associated pencil T_λ are called of *general type* if the following two nondegeneracy requirements are satisfied:

- (i) $\deg Q_k(z) = k$ (i.e., $a_{k,k} \neq 0$),
- (ii) no two roots of the (characteristic) equation

$$a_{k,k} + a_{k-1,k-1}t + \dots + a_{0,0}t^k = 0 \quad (1.3)$$

lie on a line through the origin (in particular, 0 is not a root of (1.3)).

The first statement of the paper is as follows.

Proposition 1. *If the characteristic equation (1.3) has k distinct solutions $\alpha_1, \alpha_2, \dots, \alpha_k$ and satisfies the preceding nondegeneracy assumptions (in particular, these imply that $a_{0,0} \neq 0$ and $a_{k,k} \neq 0$) then*

- (i) *for all sufficiently large n there exist exactly k distinct eigenvalues $\lambda_{n,j}$, $j = 1, \dots, k$, such that the associated spectral pencil T_λ has a polynomial eigenfunction $p_{n,j}(z)$ of degree exactly n ,*
- (ii) *the eigenvalues $\lambda_{n,j}$ split into k distinct families labeled by the roots of (1.3) such that the eigenvalues in the j -th family satisfy*

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n,j}}{n} = \alpha_j, \quad j = 1, \dots, k.$$

Theorem 1. *In the notation of Proposition 1 for any pencil T_λ of general type and every $j = 1, \dots, k$ there exists a subsequence $\{n_{i,j}\}$, $i = 1, 2, \dots$, such that the limits*

$$\Psi_j(z) := \lim_{i \rightarrow \infty} \frac{p'_{n_{i,j}}(z)}{\lambda_{n_{i,j}} p_{n_{i,j}}(z)}, \quad j = 1, \dots, k,$$

exist almost everywhere in \mathbb{C} and are analytic functions in some neighborhood of ∞ . Each $\Psi_j(z)$ satisfies equation (1.2), i.e., $\sum_{i=0}^k Q_i(z) \Psi_j^i(z) = 0$ almost everywhere in \mathbb{C} , and the functions $\Psi_1(z), \dots, \Psi_k(z)$ are independent sections of Γ considered as a branched covering over \mathbb{CP}^1 in a sufficiently small neighborhood of ∞ .

As we explain in §5, a key ingredient in the proof of Theorem 1 is the following localization result for the roots of the above eigenpolynomials.

Theorem 2. *For any general type pencil T_λ all the roots of all its polynomial eigenfunctions $p_{n,j}(z)$ lie in a certain disk in \mathbb{C} centered at the origin.*

The sketch of the proof of this fundamental result is as follows. We convert the differential equation satisfied by the eigenpolynomials into a non-linear Riccati type equation for the logarithmic derivatives of the eigenpolynomials. This equation is then solved recursively and the formal solution is shown to be analytic in a neighborhood of ∞ . This shows that the zeros of the eigenpolynomials (coinciding with the poles of the logarithmic derivatives) must lie in some compact subset of \mathbb{C} .

Moreover, we conjecture that for each j the above convergence result holds in fact for the whole corresponding sequence of eigenpolynomials.

Conjecture 1. *In the notation of Theorem 1, for every $j = 1, \dots, k$ the limit*

$$\Psi_j(z) = \lim_{n \rightarrow \infty} \frac{p'_{n,j}(z)}{\lambda_{n,j} p_{n,j}(z)}$$

exists and has all the properties stated in Theorem 1 almost everywhere in \mathbb{CP}^1 .

We emphasize the fact that the *proof* of Theorem 1 actually shows that Conjecture 1 is valid (at least) in a neighborhood of infinity.

In order to formulate our further results and reinterpret Theorems 1–2 and Conjecture 1 we need some notation and several basic notions of potential theory.

Notation 2. The function $L_{n,j}(z) = \frac{p'_{n,j}(z)}{\lambda_{n,j} p_{n,j}(z)}$ is called the *normalized logarithmic derivative* of the polynomial $p_{n,j}(z)$. The limit $\Psi_j(z) = \lim_{n \rightarrow \infty} L_{n,j}(z)$ (if it exists) will be referred to as the *asymptotic (normalized) logarithmic derivate* of the family $\{p_{n,j}(z)\}$.

The *root-counting measure* μ_P of a given polynomial $P(z)$ of degree m is the finite probability measure obtained by placing the mass $\frac{1}{m}$ at every root of $P(z)$. (If some root is multiple we place at this point the mass equal to its multiplicity divided by m .) Given a sequence $\{P_m(z)\}$ of polynomials we call the limit $\mu = \lim_{m \rightarrow \infty} \mu_{P_m}$ (if it exists in the sense of the weak convergence of functionals) the *asymptotic root-counting measure* of the sequence $\{P_m(z)\}$.

If μ exists and is compactly supported it is also a probability measure and its support $\text{supp } \mu$ is the limit (in the set-theoretic sense) of the sequence $\{Z_{P_m}\}$ of the zero loci to $\{P_m(z)\}$.

The Cauchy transform of a complex-valued compactly supported finite measure ρ is given by

$$C_\rho(z) = \int_{\mathbb{C}} \frac{d\rho(\xi)}{z - \xi}.$$

Note that $C_\rho(z)$ is defined at each point z for which the Newtonian potential

$$U_{|\rho|}(z) = \int_{\mathbb{C}} \frac{d|\rho|(\zeta)}{|\zeta - z|}$$

is finite. It is easy to see that $C_\rho(z)$ exists a.e. and that the original measure ρ can be restored from its Cauchy transform by the formula

$$\rho = \frac{1}{\pi} \frac{\partial C_\rho(z)}{\partial \bar{z}},$$

where $\frac{1}{\pi} \frac{\partial C_\rho(z)}{\partial \bar{z}}$ is considered as a distribution, see e.g. [16, Ch. 2].

Note also that the Cauchy transform $C_{\mu_P}(z)$ of the root-counting measure μ_P of a given degree m polynomial $P(z)$ coincides with $\frac{P'}{mP}$.

A reinterpretation of Theorem 1 in the above terms is as follows.

Proposition 2. *In the notation of Theorem 1, for each $j = 1, \dots, k$ there exists a subsequence $\{n_{i,j}\}$, $i = 1, 2, \dots$, such that the asymptotic root-counting measure μ_j of the family $\{p_{n_{i,j}}(z)\}_i$ exists and has compact support with vanishing Lebesgue area. The Cauchy transform of μ_j coincides with $\alpha_j \Psi_j$ almost everywhere in \mathbb{C} .*

As an illustration we present below some numerical results showing a rather complicated behavior of the zero loci $\mathcal{Z}_{p_{n,j}}$ for $j = 1, \dots, k$. Recall that $\text{supp } \mu_j = \lim_{n \rightarrow \infty} \mathcal{Z}_{p_{n,j}}$ and note that there are slight differences between the scalings used in the four pictures shown in Fig. 1.

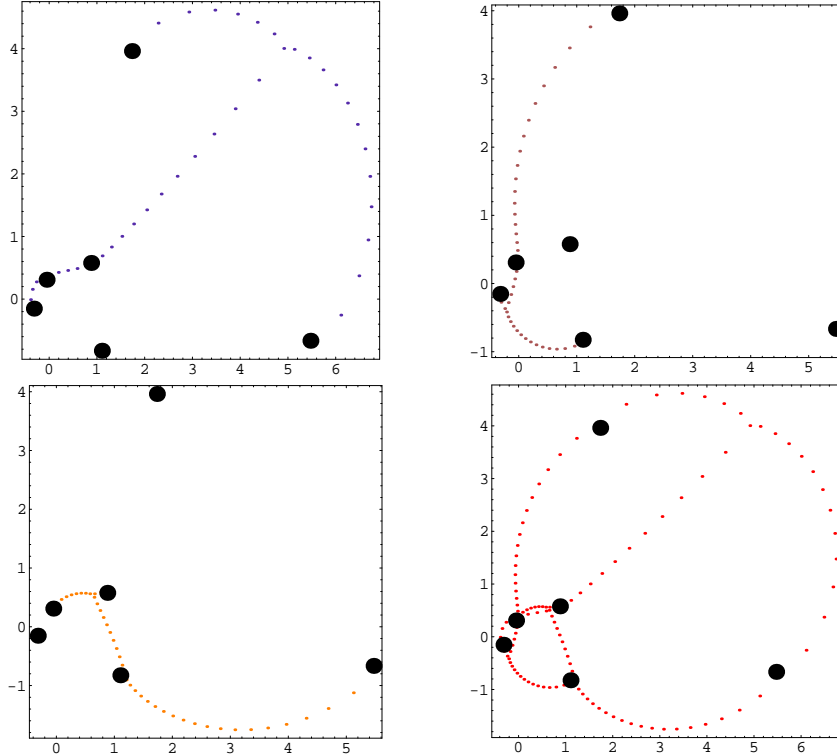


FIGURE 1. Three root-counting measures for a third order homogenized spectral pencil.

Explanations to Fig. 1. The two pictures in the upper row and the left picture in the bottom row show the roots of three eigenpolynomials of degree 55 for the

(ad hoc chosen) homogenized spectral pencil

$$T_\lambda = (z^3 - (5 + 2I)z^2 + (4 + 2I)z) \frac{d^3}{dz^3} + \lambda(z^2 + Iz + 2) \frac{d^2}{dz^2} + \lambda^2 \frac{1}{5}(z - 2 + I) \frac{d}{dz} + \lambda^3$$

consisting of ES_3 -operators. The remaining picture shows the union of the roots of all three polynomials. The six fat points on all pictures are the branching points of the associated algebraic curve Γ given by

$$(z^3 - (5 + 2I)z^2 + (4 + 2I)z) w^3 + (z^2 + Iz + 2)w^2 + \frac{1}{5}(z - 2 + I)w + 1 = 0$$

and considered as the branched covering over the z -plane. Numerical comparisons of the eigenfunctions of different degrees show that the above three root distributions already give a very good approximation of the three corresponding limiting probability measures μ_1, μ_2, μ_3 whose Cauchy transforms generate (after appropriate scalings, see Theorem 1) the three branches of Γ near ∞ . Note that all three supports end only at the six branching points of Γ .

As far as the measures μ_j are concerned, in this paper we establish just some of their most important properties. To prove these we use two facts about $C_{\mu_j}(z)$. First, that $\alpha_j^{-1} C_{\mu_j}(z)$ satisfies the algebraic equation (1.2) and hence can be locally written as a finite sum $\sum_{i=1}^r \chi_i \alpha_i^{-1} \Psi_i$, where the χ_i 's are characteristic functions of certain sets. Second, that

$$\frac{\partial C_{\mu_j}(z)}{\partial \bar{z}} \geq 0$$

in the sense of distributions. Using these facts we can develop a natural algebraic geometric setting and build on complex analytic techniques based on the main results of [10] in order to prove the next two theorems.

Theorem 3. *For each pencil T_λ of general type there exists a real-analytic subset Γ_2 of \mathbb{C} such that any limiting measure μ_j has properties (A)–(C) below in any sufficiently small neighborhood $\Omega(z_0)$ of any point $z_0 \in \mathbb{C} \setminus \Gamma_2$. In what follows Γ denotes the curve associated with the pencil T_λ , $\alpha_j \in \mathbb{C}$ is as in Proposition 1, and for any branch γ_i of Γ we let $A_i := \alpha_j \gamma_i$ (clearly, A_i depends on j).*

- (A) *The support S of the measure μ_j restricted to $\Omega(z_0)$ is a finite union of smooth curves S_r , $r \in J$.*
- (B) *For each S_r and any $\tilde{z} \in S_r$ lying in $\Omega(z_0)$ one can always choose two branches $\gamma_1(z)$ and $\gamma_2(z)$ of Γ such that the tangent line $l(\tilde{z})$ to S_r is orthogonal to $A_1(\tilde{z}) - A_2(\tilde{z})$.*
- (C) *The density of the measure μ_j at \tilde{z} equals $\frac{|A_1(\tilde{z}) - A_2(\tilde{z})| ds}{2\pi}$, where ds is the length element along the curve S_r .*

In fact for most pencils we can do better:

Theorem 4. *For a typical general type pencil T_λ the set Γ_2 in the preceding theorem is finite.*

Remark 1. In the case of the usual spectral problem strong results in this direction were obtained in [4].

We note that Theorem 1 also leads to a partial progress on the following intriguing question in potential theory. Let $Alg(i, j)$ denote the linear space of all polynomials in the variables (y, z) of bidegrees at most (i, j) , i.e., each $P(y, z) \in Alg(i, j)$ has the form $\sum_{l=0}^i P_l(z) y^l$, where $\deg P_l(z) \leq j$, $l = 1, \dots, i$. Abusing the notation we identify each $P(y, z) \not\equiv 0$ with the algebraic function in the variable z defined by the equation $P(y, z) = 0$.

Problem 2. Describe the subset $PAlg(i, j) \subset Alg(i, j)$ consisting of all algebraic functions which have a branch near ∞ coinciding with the Cauchy transform of some probability measure compactly supported in \mathbb{C} .

We refer to such algebraic functions as *positive Cauchy transforms*. The only case when a complete answer to the above problem seems to be obvious is for bidegrees $(1, j)$, i.e., the case of rational functions. Namely, a rational function $r(z) = \frac{p(z)}{q(z)}$ is a positive Cauchy transform if and only if $r(z)$ is of the form

$$\sum_{l=1}^j \frac{c_l}{z - z_l} \text{ with } c_l > 0, 1 \leq l \leq j, z_k \neq z_l \text{ for } k \neq l, \sum_{l=1}^j c_l = 1.$$

Let us finally give yet another interpretation of Theorem 1 and Proposition 2.

Corollary 1. Each branch near infinity of an algebraic function satisfying (1.2) is a positive Cauchy transform multiplied by an appropriate constant.

The structure of the paper is as follows. In §2 we study the asymptotics of the eigenvalues to (1.1) and some simple properties of the corresponding eigenfunctions as well as the defining algebraic curve (1.2). In §3 we solve the spectral problem defined by the operator (1.1) in formal power series near ∞ using the variable $y := z^{-1}$. In §4 we show that the power series solution obtained in §3 converges in some neighborhood of ∞ . Based on these results we prove Theorem 2 as well as Theorem 1 and its corollaries in §5. In §6 we prove Theorem 3 and complete the proof of Theorem 4. Finally, in §7 we propose a number of open problems and conjectures on the asymptotic behavior of polynomial functions for linear ordinary differential operators and place these in a wider context that encompasses both old and new literature on this and related topics.

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2. BASIC FACTS ON ASYMPTOTICS OF EIGENVALUES AND POLYNOMIAL EIGENFUNCTIONS

In this section we prove Proposition 1, that is, we describe the polynomial solutions of an ES spectral pencil and we also prove the easy part of Theorem 1 saying that if the limit functions $\Psi_j(z)$, $1 \leq j \leq k$, exist locally and have derivatives of sufficiently high order then they must satisfy equation (1.2).

Denote by $\mathcal{D}_n \subset ES_k$ the subset of all ES -operators T of order at most k such that the equation $Ty = 0$ has a polynomial solution of degree exactly n .

Lemma 1. In the notation of Theorem 1 the closure $\overline{\mathcal{D}_n}$ of the discriminant \mathcal{D}_n is a hyperplane in ES_k given by the equation

$$\sum_{i=0}^k n(n-1) \dots (n-i+1) a_{i,i} = 0. \quad (2.1)$$

Proof. Since any ES -operator T (nonstrictly) preserves the infinite flag $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \dots \subset \dots$ of linear spaces of polynomials of at most given degree it follows that T has an eigenpolynomial of degree exactly n if and only if

- (i) T restricted to \mathcal{P}_n is degenerate,
- (ii) the kernel of T restricted to \mathcal{P}_n intersects $\mathcal{P}_n \setminus \mathcal{P}_{n-1}$.

The closure $\overline{\mathcal{D}_n}$ consists of all T having an eigenpolynomial of degree at most n . Now T is upper triangular in the standard monomial basis and the j -th diagonal entry of T equals $\sum_{i=0}^k j(j-1)\dots(j-i+1)a_{i,i}$. Therefore $\overline{\mathcal{D}_n}$ is given by equation (2.1). \square

Remark 2. If $T \in \overline{\mathcal{D}_n}$ but $T \notin \overline{\mathcal{D}_l}$ for $0 \leq l < n$ then T has a polynomial solution of degree exactly n . Otherwise T has at least a polynomial solution of degree equal to $\min\{l \in \{0, \dots, n-1\} : T \in \overline{\mathcal{D}_l}\}$ (and probably some other polynomial solutions as well).

We now turn to Proposition 1.

Proof. By Lemma 1 the homogenized spectral pencil T_λ has a polynomial solution of degree at most n if

$$\sum_{i=0}^k n(n-1)\dots(n-i+1)a_{i,i}\lambda^{k-i} = 0. \quad (2.2)$$

This equation is of degree exactly k in λ since $a_{0,0} \neq 0$. Considering n for the moment as a complex variable, the equation defines an algebraic curve in \mathbb{C}^2 with k branches over \mathbb{C} . The behavior of the branches at infinity may be found by substituting $\tilde{\lambda} = \lambda/n$ and dividing the left-hand side of (2.2) by n^k , which gives

$$\sum_{i=0}^k 1 \cdot \frac{(n-1)}{n} \cdot \frac{(n-2)}{n} \cdot \dots \cdot \frac{(n-i+1)}{n} \cdot a_{i,i} \cdot \tilde{\lambda}^{k-i} = 0. \quad (2.3)$$

If $n \rightarrow \infty$ the latter family of equations tends coefficientwise to equation (1.3). Since the latter has exactly k different solutions, this is true for (2.3) as well if $n \in N$ for some simply connected neighborhood $N \subset \mathbb{CP}^1$ of infinity. Hence in N there are k different branches $\tilde{\lambda}_{n,1}, \dots, \tilde{\lambda}_{n,k}$ of the algebraic function $\tilde{\lambda}$ defined as a function of the complex variable n by equation (2.3), and corresponding branches $\lambda_{n,1}, \dots, \lambda_{n,k}$ of the algebraic function λ . Then clearly one has

$$\lim_{n \rightarrow \infty} \lambda_{n,j}/n = \lim_{n \rightarrow \infty} \tilde{\lambda}_{n,j} = \alpha_j, \quad j = 1, \dots, k.$$

This completes the proof of part (ii). To prove further that for each branch $\lambda_{n,j}$, $j = 1, \dots, k$, the associated spectral pencil T_λ has a polynomial eigenfunction $p_{n,j}(z)$ of degree *exactly* n , we need to show that for a λ and n that solve equation (2.2) there is no $m \neq n$ such that λ solves equation (2.2) for that m . This follows if we first prove that there are no solutions $n \neq m$ to $\lambda_{n,j} = \lambda_{m,j}$, $j = 1, \dots, k$, and secondly that if $j_1 \neq j_2$ then there are no solutions to $\lambda_{n,j_1} = \lambda_{m,j_2}$ (both statements in a possibly shrunk neighborhood of infinity). The first statement is an immediate consequence of the assumptions since they imply that $\alpha_j \neq 0$, $j = 1, \dots, k$, and thus for each j the function $n \mapsto \lambda_{n,j}$ is one-to-one in a neighborhood of infinity. For the second statement argue by contradiction as follows. If the statement is false there are unbounded sequences of positive integers n_i, m_i , $i = 1, 2, \dots$ such that $\lambda_{n_i,j_1} = \lambda_{m_i,j_2}$ for all i . We may assume (by choosing a subsequence and possibly interchanging j_1 and j_2) that $\lim_{i \rightarrow \infty} m_i/n_i = r \in \mathbb{R}$. Hence

$$\alpha_{j_1} = \lim_{i \rightarrow \infty} \lambda_{n_i,j_1}/n_i = \lim_{i \rightarrow \infty} (\lambda_{m_i,j_2}/m_i)(m_i/n_i) = r\alpha_{j_2}$$

and so the arguments of α_{j_1} and α_{j_2} are equal, which contradicts the nondegeneracy assumptions in the proposition. This completes the proof. \square

Remark 3. It is possible and straightforward to calculate the asymptotics for the eigenvalues even in the case when (1.3) has multiple or vanishing roots.

Let us now prove the easy part of Theorem 1.

Proposition 3. *Let $\{p_{n,j}(z)\}$ be a family of polynomial eigenfunctions of a homogenized spectral pencil T_λ with corresponding family of eigenvalues $\{\lambda_{n,j}\}$, i.e., $p_{n,j}(z)$ satisfies the equation $T_{\lambda_{n,j}}p_{n,j}(z) = 0$. Assume that the following holds:*

- (i) $\lim_{n \rightarrow \infty} \lambda_{n,j} = \infty$,
- (ii) *there exists an open set $\Omega \subseteq \mathbb{CP}^1$ where the normalized logarithmic derivatives $L_{n,j}(z) = \frac{p'_{n,j}(z)}{\lambda_{n,j} p_{n,j}(z)}$ are defined for all sufficiently large n and the sequence $\{L_{n,j}(z)\}$ converges in Ω to a function*

$$\Psi_j(z) := \lim_{n \rightarrow \infty} L_{n,j}(z),$$

- (iii) *the $k-1$ first derivatives of the sequence $\{L_{n,j}(z)\}$ are uniformly bounded in Ω .*

Then if $\Psi_j(z)$ does not vanish identically it satisfies the equation

$$\sum_{i=0}^k Q_i(z) \Psi_j^i(z) = 0. \quad (2.4)$$

Proof. In order to simplify the notation in this proof let us fix the value of the index $j \in \{1, \dots, k\}$ and simply drop it.

Note that each $L_n(z) = \frac{p'_n(z)}{\lambda_n p_n(z)}$ is well defined and analytic in any disk \mathbf{D} free from the zeros of $p_n(z)$. Choosing such a disk \mathbf{D} and an appropriate branch of the logarithm such that $\log p_n(z)$ is defined in \mathbf{D} let us consider a primitive function $M(z) = \lambda_n^{-1} \log p_n(z)$ which is also well defined and analytic in \mathbf{D} .

Straightforward calculations give: $e^{\lambda_n M(z)} = p_n(z)$, $p'_n(z) = p_n(z) \lambda_n L_n(z)$, and $p''_n(z) = p_n(z) (\lambda_n^2 L_n^2(z) + \lambda_n L'_n(z))$. More generally,

$$\frac{d^i}{dz^i} (p_n(z)) p_n(z) \left(\lambda_n^i L_n^i(z) + \lambda_n^{i-1} F_i(L_n(z), L'_n(z), \dots, L_n^{(i-1)}(z)) \right),$$

where the second term

$$\lambda_n^{i-1} F_i(L_n, L'_n, \dots, L_n^{(i-1)}) \quad (2.5)$$

is a polynomial of degree $i-1$ in λ_n . The equation $T_{\lambda_n} p_n(z) = 0$ gives us

$$p_n(z) \left(\sum_{i=0}^k Q_i(z) \lambda_n^{k-i} \left(\lambda_n^i L_n^i(z) + \lambda_n^{i-1} F_i(L_n(z), L'_n(z), \dots, L_n^{(i-1)}(z)) \right) \right) = 0$$

or equivalently,

$$\lambda_n^k \sum_{i=0}^k Q_i(z) \left(L_n^i(z) + \lambda_n^{-1} F_i(L_n(z), L'_n(z), \dots, L_n^{(i-1)}(z)) \right) = 0. \quad (2.6)$$

Letting $n \rightarrow \infty$ and using the boundedness assumption for the first $k-1$ derivatives we get the required equation (2.4). \square

We end this section by establishing an important property of the curve Γ given by (1.2). Note that unless $Q_k(x) \equiv 0$ the curve Γ is a k -sheeted branched covering of the z -plane. We want to describe the behavior of Γ at infinity. Using a change of coordinate $y := z^{-1}$ we can rewrite equation (1.2) as

$$\sum_{i=0}^k Q_i(z) w^i = \sum_{i=0}^k z^{-i} Q_i(z) (wz)^i = \sum_{i=0}^k P_i(y) \xi^i = 0,$$

where

$$P_i(y) = z^{-i} Q_i(z) = \sum_{j=0}^i a_{ij} y^{i-j}$$

and $\xi := wz = w/y$. Note also that at the point $y = 0$ one gets the reciprocal characteristic equation

$$a_{k,k}\xi^k + a_{k-1,k-1}\xi^{k-1} + \dots + a_{0,0} = 0. \quad (2.7)$$

Remark 4. The roots ξ_1, \dots, ξ_k of (2.7) are the inverses of the roots $\alpha_1, \dots, \alpha_k$, respectively, of (1.3).

Using the above argument we get the following simple statement.

Lemma 2. *If the roots ξ_1, \dots, ξ_k of equation (2.7) are pairwise distinct then there are k branches $\gamma_i(z)$, $i = 1, \dots, k$, of the curve Γ that are well defined in some common neighborhood of $z = \infty$. The i -th branch $\gamma_i(z)$ satisfies the normalization condition $\lim_{z \rightarrow \infty} z\gamma_i(z) = \xi_i$.*

3. SOLVING EQUATION (2.6) FORMALLY

This and the following two sections are completely devoted to the proof of Theorems 1 and 2. The sketch of the proof of Theorem 1 is as follows. In Proposition 3 of §2 we have transformed the linear differential equation for the eigenpolynomials into a non-linear Riccati type equation for the logarithmic derivative. In this section we first analyze closely the terms of this new equation. We then describe the recursion scheme for solving this equation formally in a neighborhood of infinity and see how the solutions behave when $\lambda \rightarrow \infty$. Finally, in the next section we show that there is a neighborhood of $z = \infty$ where the formal solutions are analytic and we complete the proofs of Theorems 1 and 2 in §5.

Throughout this section we use the variable $y := 1/z$ near $z = \infty$.

The differential algebra $A_{z,L}$. As the *first step* we describe the terms occurring in equation (2.6) more precisely. It is convenient to do this in a universal setting using the following infinitely generated free commutative \mathbb{C} -algebra (or rather the free differential algebra, cf. [20])

$$A_{z,L} := \mathbb{C}[\lambda, \lambda^{-1}, z, z^{-1}, L^{(0)}, L^{(1)}, L^{(2)}, \dots].$$

This algebra should be thought of as (a universal object) containing the terms in equation (2.6). Concrete instances can be obtained by specialization. In particular, the $L^{(i)}$'s correspond to the normalized logarithmic derivatives in (2.6).

Note that the monomials $\lambda^i z^j L^I$ form a basis of $A_{z,L}$ considered as a vector space, where for any multi-index $I = (i_1, \dots, i_r)$ the symbol L^I denotes the product

$$L^I = \prod_{s=1}^r L^{(i_s)}.$$

It suffices to use multi-indices I that are finite non-decreasing sequences of non-negative integers. Denote the set of all such multi-indices by FS . (By definition, $L^{(0)} := L$.) Such index sequences may alternatively be thought of as finite multisets. In particular, we include the empty sequence \emptyset in FS and interpret $L^\emptyset := 1$. For a given multi-index I define its *modulus* by $|I| = \sum_{s=1}^r i_s$ and its *length* by $lng(I) = r$. (By definition, $|\emptyset| := 0$ and $lng(\emptyset) := 0$). For a given $I \in FS$ denote by I_+ the sequence obtained from I by discarding all its 0 elements.

$A_{z,L}$ is equipped with a natural derivation D_z which is a prototype of $\frac{d}{dz}$. Namely, D_z is uniquely defined by the relations: $D_z * \lambda = 0$, $D_z * z = 1$ and $D_z * L^{(i)} = L^{(i+1)}$. (We use the symbol “ $*$ ” to denote the action of a differential operator as opposed to the product of differential operators. Note that the ring of differential operators generated by $A_{z,L}$ and D_z acts on $A_{z,L}$.)

The normalized logarithmic derivative in the universal setting. We can next use $A_{z,L}$ to describe the relation between the derivatives of an eigenpolynomial

and its normalized logarithmic derivative. This is done by defining a differential $A_{z,L}$ -module. Consider the free rank one $A_{z,L}$ -module $A_{z,L}e^M$, where e^M is the generator. Define $D_z * e^M := \lambda L e^M$ and extend the action of D_z to the whole $A_{z,L}e^M$ using the Leibnitz rule. Note that this action intuitively just says that L is the normalized logarithmic derivative $D_z * e^M / \lambda e^M$ of the generator e^M . The following lemma can be easily proved by induction.

Lemma 3. *In the above notation one has $(D_z)^i * e^M = R_i e^M$, where $R_0 = 1$, $R_1 = \lambda L$ and $R_{i+1} = (\lambda L + D_z) * R_i$ for $i \geq 1$. In other words,*

$$R_i = (\lambda L + D_z)^i * 1 = (\lambda L + D_z)^{i-1} * \lambda L, \quad i \geq 1.$$

The R_i considered as polynomials $R_i(\lambda, L^{(0)}, \dots)$ have (by universality) the following property.

Lemma 4. *Let $g = \lambda^{-1} D \log f$, where f is a non-vanishing analytic function in an open subset $\Omega \subset \mathbb{C}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ (so g is the logarithmic derivative of f normalized with respect to λ). Then in the above notation one has*

$$f^{(i)} = R_i(\lambda, g, g^{(1)}, \dots) f, \quad i \geq 1.$$

The differential algebra $A_{z,L}$ at ∞ . Next we rewrite (2.6) using the variable $y = z^{-1}$. Note that if $p(z)$ is a non-constant polynomial then for any $\kappa \neq 0$ the logarithmic derivative $L(z) = p'(z)/\kappa p(z)$, rewritten using the variable $y = \frac{1}{z}$, has a simple zero at $y = 0$. Hence we should look for solutions of equation (2.6) in the form $L(z) = yN(y)$.

First we describe $A_{z,L}$ near $z = \infty$. For this we define an analogous free commutative algebra

$$B_{y,N} := \mathbb{C}[\lambda, \lambda^{-1}, y, y^{-1}, N^{(0)}, N^{(1)}, \dots].$$

As above, it has a natural derivation D_y which is a prototype of $\frac{d}{dy}$ satisfying the relations: $D_y * \lambda = 0$, $D_y * y = 1$ and $D_y * N^{(i)} = N^{(i+1)}$. Note that $\frac{d}{dz} = -y^2 \frac{d}{dy}$ and recall that $L := L^{(0)}$ and $N := N^{(0)}$. Define an injection $\Theta : A_{z,L} \rightarrow B_{y,N}$ of algebras determined by

$$\Theta(\lambda) = \lambda, \quad \Theta(z) = y^{-1}, \quad \Theta(L) = yN, \quad \Theta(L^{(i)}) = (-y^2 D_y)^i * yN.$$

The following lemma describes the connection between $A_{z,L}$ and $B_{y,N}$.

Lemma 5. *The injection Θ has the property that for any $a \in A_{z,L}$ one has*

$$\Theta(D_z^i * a) = (-y^2 D_y)^i * \Theta(a).$$

Proof. By definition the above formula is valid in the case $i = 1$ for all the generators of $A_{z,L}$. Since D_z and $-y^2 D_y$ are derivations the formula then works for all elements in this algebra and $i = 1$. Simple induction shows that the above formula is valid even for all $i > 1$. \square

Main lemma on R_i . The preceding lemmas imply the following relation:

$$\lambda^{-i} y^{-i} \Theta(R_i) = y^{-i} (yN - y^2 \lambda^{-1} D_y)^i * 1 = y^{-i} (yN - y^2 \lambda^{-1} D_y)^{i-1} y * N. \quad (3.1)$$

We need to know which monomial terms of the form $\lambda^{\alpha_1} y^{\alpha_2} N^I$ occur in this equation. These monomials are contained in the subalgebra $B_0 \subset B_{y,N}$ defined as

$$B_0 := \mathbb{C}[\lambda^{-1}, y, N, \lambda^{-1} y N^{(1)}, \dots, (\lambda^{-1} y)^j N^{(j)}, \dots],$$

see Lemma 6 below. Define a (necessarily two-sided, by commutativity) ideal $J \subset B_0$ as follows:

$$J := \langle \lambda^{-1}, (\lambda^{-1} y)^2 N^{(1)} N^{(1)}, \dots, (\lambda^{-1} y)^{j_1+j_2} N^{(j_1)} N^{(j_2)} \dots \rangle,$$

where $j_1 \geq 1$, $j_2 \geq 1$. (As we will see the ideal J contains all the terms that do not influence the asymptotic behavior of the equation (2.6).)

One can easily show that the set of all monomials of the form

$$(\lambda^{-1})^{\alpha_1} y^{\alpha_2} (\lambda^{-1} y)^{|I|} N^I,$$

where $\alpha_i \in \mathbb{Z}_{\geq 0}$, $i = 1, 2$, and $I \in FS$ constitutes a basis of B_0 as a vector space. Such a monomial belongs to J if and only if either $\alpha_1 \geq 1$ or $\text{lng}(I_+) \geq 2$.

Lemma 6. *For all $i \geq 0$ the following identity between elements in B_0 holds:*

$$\lambda^{-i} y^{-i} \Theta(R_i) = N^i + \sum_{j=1}^{i-1} \binom{i}{j+1} (-1)^j (\lambda^{-1} y)^j N^{i-1-j} N^{(j)} + h \quad (3.2)$$

for some unique $h \in J$. Moreover, if $i \geq 1$, the exponents of all non-vanishing monomials $(\lambda^{-1})^{\alpha_1} y^{\alpha_2} (\lambda^{-1} y)^{|I|} N^I$ occurring in the right-hand side of (3.2) satisfy the inequality $\alpha_1 + |I| \leq i - 1$.

Proof. The case $i = 0$ is trivial so we assume that $i \geq 1$.

Claim. The differential operator $L_i := y^{-(i+1)}(yN - y^2 \lambda^{-1} D_y)^i y$ preserves both B_0 and J and satisfies the relation

$$L_i * N = N^{i+1} + \sum_{j=1}^i \binom{i+1}{j+1} (-1)^j N^{i-j} (\lambda^{-1} y)^j N^{(j)} \pmod{J}, \quad (3.3)$$

where \pmod{J} means that the expression is considered modulo the ideal $J \subset B_0$.

The lemma immediately follows from this claim. Indeed, note that by Lemma 3 one has

$$\lambda^{-i} y^{-i} \Theta(R_i) = y^{-i} (yN - y^2 \lambda^{-1} D_y)^{i-1} y * N = L_{i-1} * N \quad (3.4)$$

and so (3.2) is a consequence of (3.4) and (3.3). It thus remains to prove the three assertions in the above claim, which we do below by induction. As the base of induction note that since $L_0 = 1$ it is obvious that L_0 preserves both B_0 and J and that it satisfies (3.3).

Induction steps. We first show inductively that all operators L_i preserve both B_0 and J and then using this we check formula (3.3) again by induction. Now the following identity is immediate:

$$\begin{aligned} L_{i+1} * v &= y^{-(i+2)} (yN - y^2 \lambda^{-1} D_y) y^{i+1} L_i * v = \\ &= (N - (i+1) \lambda^{-1}) L_i * v - y \lambda^{-1} D_y (L_i * v), \quad v \in B_0. \end{aligned} \quad (3.5)$$

Hence in order to check that B_0 and J are preserved by L_{i+1} it suffices (using the induction assumption) to show that both $(N - (i+1) \lambda^{-1})$ and $y \lambda^{-1} D_y$ preserve B_0 and J . Since N and λ^{-1} are contained in B_0 , it is clear that they preserve both B_0 and J , so it is enough to verify that the same holds for $\lambda^{-1} y D_y$. Let us first show that the latter operator preserves the ideal J , under the assumption that it preserves B_0 . For this we note that an arbitrary element $h \in J$ may be written as $h = \sum b_i h_i$, where h_i are the generators given in the definition of J and $b_i \in B_0$. Thus it is enough to prove that $y \lambda^{-1} D_y * b_i h_i \in J$. As we will now explain, this property follows from the identity

$$y \lambda^{-1} D_y * b_i h_i = y \lambda^{-1} h_i (D_y * b_i) + y \lambda^{-1} b_i (D_y * h_i).$$

Indeed, we claim that its right-hand side belongs to J . To show this note that the first term clearly belongs to J , so it suffices to check that $y \lambda^{-1} (D_y * h_i) \in J$. Now

if $h_i = \lambda^{-1}$ then this is again obvious. For the other generators we simply use the identity

$$\begin{aligned} y\lambda^{-1}D_y * (\lambda^{-1}y)^{j_1+j_2}N^{(j_1)}N^{(j_2)} &= (j_1+j_2)\lambda^{-1}(\lambda^{-1}y)^{j_1+j_2}N^{(j_1)}N^{(j_2)} + \\ &+ (\lambda^{-1}y)^{j_1+j_2+1}N^{(j_1+1)}N^{(j_2)} + (\lambda^{-1}y)^{j_1+j_2+1}N^{(j_1)}N^{(j_2+1)}. \end{aligned}$$

The terms in the right-hand side of the latter formula clearly belong to J and so from (3.5) we get that L_{i+1} preserves J as soon as L_i does. The fact that $\lambda^{-1}yD_y$ preserves B_0 is proved in exactly the same fashion, namely by first checking that this is true for the generators of B_0 and then applying an inductive argument based on the observation that $\lambda^{-1}yD_y$ is a derivation.

Now that we established that both B_0 and J are preserved by the operators L_i we use this information to check the induction step in formula (3.3). Since $\lambda^{-1} \in J$, equation (3.5) taken modulo J gives

$$L_{i+1} * v = (N - (i+1)\lambda^{-1})L_i * v - y\lambda^{-1}D_y(L_i * v) = NL_i * v - y\lambda^{-1}D_y(L_i * v) \quad (3.6)$$

for $v \in B_0$. Using the relations: $y\lambda^{-1}D_y * N^{i+1} = (i+1)y\lambda^{-1}N^iN^{(1)}$ and

$$y\lambda^{-1}D_y * N^{i-j}y^j\lambda^{-j}N^{(j)} = N^{i-j}y^{j+1}\lambda^{-(j+1)}N^{(j+1)} \pmod{J}$$

we get from (3.6) and the induction assumption that

$$\begin{aligned} L_{i+1} * N &= NL_i * N - y\lambda^{-1}D_y * (L_i * N) = \\ &N^{i+2} + \sum_{j=1}^i \binom{i+1}{j+1} (-1)^j N^{i+1-j} y^j \lambda^{-j} N^{(j)} - (i+1)y\lambda^{-1}N^iN^{(1)} - \\ &\sum_{j=1}^i \binom{i+1}{j+1} (-1)^j N^{i-j} y^{j+1} \lambda^{-(j+1)} N^{(j+1)} \pmod{J}. \end{aligned}$$

The usual properties of binomial coefficients now accomplish the proof of the step of induction, which completes the proof of formula (3.3).

The last statement in Lemma 6 follows trivially since the degree of the variable λ^{-1} in $\lambda^{-i}y^{-i}\Theta(R_i)$ (considered as an element of $B_{y,N}$) is precisely $i-1$, see (3.4), and the degree in λ^{-1} of the monomial $(\lambda^{-1})^{\alpha_1}y^{\alpha_2}(\lambda^{-1}y)^{|I|}N^I$ equals $\alpha_1 + |I|$. \square

Description of the terms in equation (2.6). Consider now the homogenized spectral pencil $T_\lambda = \sum_{i=0}^k Q_i(z)\lambda^{k-i}\frac{d^i}{dz^i}$. It acts (if we substitute D_z^i for $\frac{d^i}{dz^i}$) on the differential module $A_{z,L}e^M$ and satisfies the obvious relation

$$T_\lambda * e^M = \sum_{i=0}^k Q_i(z)\lambda^{k-i}R_i e^M.$$

Recall from Lemma 3 that each R_i is a polynomial $R_i(\lambda, L^{(0)}, \dots)$. Now the non-linear equation $\sum_{i=0}^k Q_i(z)\lambda^{k-i}R_i = 0$ in L is the analog over $A_{z,L}$ of the non-linear equation (2.6). A change of variables $y = z^{-1}$ and division by λ^k transforms the previous equation into the equation

$$\lambda^{-k}S_\lambda(yN) := \Theta\left(\sum_{i=0}^k Q_i(z)\lambda^{-i}R_i\right) = 0. \quad (3.7)$$

In what follows we use the notation

$$\begin{aligned} \Theta(Q_i(z)) &= \Theta(a_{i,i}z^i + a_{i,i-1}z^{i-1} + \dots + a_{i,0}) = \\ &y^{-i}(a_{i,i} + a_{i,i-1}y + \dots + a_{i,0}y^i) = y^{-i}P_i(y), \end{aligned}$$

where $P_i(y)$ is a polynomial in y of degree at most i . Equation (3.7) then takes the form

$$\lambda^{-k} S_\lambda(yN) := \sum_{i=0}^k P_i(y) y^{-i} \lambda^{-i} \Theta(R_i) = 0. \quad (3.8)$$

Lemma 6 leads to the following statement.

Lemma 7. *In the above notation one has*

$$\lambda^{-k} S_\lambda(yN) = \sum_i P_i(y) N^i + \sum_{i=0}^k \sum_{j=1}^{i-1} P_i(y) \binom{i}{j+1} (-1)^j N^{i-1-j} y^j \lambda^{-j} N^{(j)} + b \quad (3.9)$$

for some $b \in J$. The exponents of all non-zero monomials $(\lambda^{-1})^{\alpha_1} y^{\alpha_2} (\lambda^{-1} y)^{|I|} N^I$ occurring in the right-hand side satisfy the condition $\alpha_1 + |I| \leq k - 1$.

Generalities on power series. Lemma 7 tells us all we need to know about the explicit form of the terms in equation (2.6). The *second step* in solving (2.6) is to find a recurrence relation for the formal power series solutions of (3.8) using (3.9). To do this we will use the Ansatz $L(y) = yN(y) = \sum_{i=1}^{\infty} \epsilon_i y^i$. In algebraic terms this means that instead of an element of $B_{y,N}$ we consider its image under the map of differential rings $B_{y,N} \rightarrow \mathbf{C}[\epsilon_1, \epsilon_2, \dots][[y]]$ given by $N \mapsto \sum_{i=0}^{\infty} \epsilon_{i+1} y^i$. (The latter ring is equipped with the usual derivation $\frac{d}{dy}$; in particular, $\frac{d}{dy} * \epsilon_i = 0$.)

We will repeatedly use some easily verified algebraic properties of the derivatives of formal power series similar to $N(y)$. These properties are summed up below.

Notation 3. If $A = \sum_{i=0}^{\infty} k_i y^i \in K[[y]]$ is a formal power series with coefficients in a field K we define $[A]_m := k_m$. The notation “ $EXP \bmod(\epsilon_1, \dots, \epsilon_m)$ ” means that expression EXP is taken modulo the vector space generated by all monomials in the indicated variables $(\epsilon_1, \dots, \epsilon_m)$. (Note that this usage is different from the earlier used “ $(\bmod J)$ ”, where J is some ideal.)

Lemma 8. *Let $N(y) = \sum_{i=0}^{\infty} \epsilon_{i+1} y^i \in \mathbf{C}[\epsilon_1, \epsilon_2, \dots][[y]]$. Then the following relations hold:*

- (1) $[y^i N^{(i)}]_m = m(m-1) \dots (m-i+1) \epsilon_{m+1}$. If $0 \leq m < i$, then $[y^i N^{(i)}]_m = 0$.
- (2) One has

$$[y^{|J|} N^J]_m = \sum_{c_1 + \dots + c_s = m} \prod_{i=1}^s c_i (c_i - 1) \dots (c_i - j_i + 1) \epsilon_{c_i+1}.$$

(If $j_i = 0$ then by abuse of notation we interpret $c_i(c_i - 1) \dots (c_i - j_i + 1)$ as 1). If J contains strictly more than one non-zero element then $[y^{|J|} N^J]_m = 0 \bmod(\epsilon_1, \dots, \epsilon_m)$.

- (3) $[N^j]_m = j \epsilon_1^{j-1} \epsilon_{m+1} \bmod(\epsilon_1, \dots, \epsilon_m)$ if $m \geq 1$ while $[N^j]_0 = \epsilon_1^j$.
- (4) $[y^i N^j N^{(i)}]_m = m(m-1) \dots (m-i+1) \epsilon_1^j \epsilon_{m+1} \bmod(\epsilon_1, \dots, \epsilon_m)$, if $i \geq 1$.
- (5) If the monomial $M = (\lambda^{-1})^{\alpha_1} y^{\alpha_2} (\lambda^{-1} y)^{|I|} N^I$ belongs to B_0 then

$$[M]_m = B(\epsilon_1, \dots, \epsilon_{m+1}, \lambda^{-1}, \lambda^{-1} m),$$

where $B(\epsilon_1, \dots, \epsilon_{m+1}, x, y)$ is a polynomial of degree at most $|I|$ in y and at most α_1 in x .

Proof. We prove properties (2) and (5) leaving the rest as an exercise for the interested reader. Note first that

$$[y^{|J|} N^J]_m = \sum_{c_1 + \dots + c_s = m} \prod_{i=1}^s [y^{j_i} N^{j_i}]_{c_i}.$$

By property (1), this gives the required relation in (2). Observe then that the only case when one has

$$\prod_{i=1}^s c_i(c_i - 1) \dots (c_i - j_i + 1) \epsilon_{c_i+1} \neq 0 \pmod{(\epsilon_1, \dots, \epsilon_m)},$$

is if some $\epsilon_{c_i+1} = \epsilon_{m+1}$, and then $c_i = m$, and all other $c_l = 0$, $l \neq i$. But if for some such $l \neq i$ we have $j_l > 0$, then $c_l(c_l - 1) \dots (c_l - j_l + 1) = 0$; hence at most one $j_i \neq 0$, $1 \leq i \leq s$. This finishes the proof of (2).

In order to prove (5) notice that (1) implies that the expression

$$A_i(\epsilon_{m+1}, \lambda^{-1}, \lambda^{-1}m) := [(\lambda^{-1}y)^i N^{(i)}]_m = \lambda^{-i}m(m-1) \dots (m-i+1)\epsilon_{m+1} = (\lambda^{-1}m)(\lambda^{-1}m - \lambda^{-1}) \dots (\lambda^{-1}m - (i-1)\lambda^{-1})\epsilon_{m+1}$$

is a polynomial whose degree in the last variable equals i . Therefore, the expression

$$A_i(\epsilon_1, \dots, \epsilon_{m+1}, \lambda^{-1}, \lambda^{-1}m) := [(\lambda^{-1}y)^{|J|} N^J]_m = \sum_{c_1 + \dots + c_s = m} \prod_{i=1}^s [(\lambda^{-1}y)^{j_i} N^{j_i}]_{c_i}$$

is a polynomial whose degree in the last variable is at most $|J|$. This implies (5). \square

The following description of the coefficients of the power series in the ideal J is an immediate consequence of Lemma 8.

Lemma 9. *Let $M = (\lambda^{-1})^{\alpha_1} y^{\alpha_2} (\lambda^{-1}y)^{|I|} N^I$, where $\alpha_i \in \mathbb{Z}_{\geq 0}$, $i = 1, 2$, $I \in FS$, be a monomial in B_0 . Then*

- (1) $[M]_0 \neq 0$ if and only if $M = (\lambda^{-1})^{\alpha_1} N^j$ for some j . If in addition $M \in J$ then $\alpha_1 \geq 1$.
- (2) If $M \in J$ and $m \geq 1$ then $[M]_m \neq 0 \pmod{(\epsilon_1, \dots, \epsilon_m)}$ implies that:
 - (a) $\alpha_1 \geq 1$,
 - (b) $\alpha_2 = 0$,
 - (c) $I = (0, \dots, 0, i)$ with $\text{lng}(I) = j+1$ and contains at most one non-zero element i .

If $i \geq 1$ then $M = (\lambda^{-1})^{\alpha_1} (\lambda^{-1}y)^i N^j N^{(i)}$ and

$$[M]_m = (\lambda^{-1})^{\alpha_1+i} m(m-1) \dots (m-i+1) \epsilon_1^j \epsilon_{m+1} \pmod{(\epsilon_1, \dots, \epsilon_m)}.$$

- (3) If $b \in J$ then $[b]_m = \lambda^{-1} E(\epsilon_1, \lambda^{-1}, \lambda^{-1}m) \epsilon_{m+1} \pmod{(\epsilon_1, \dots, \epsilon_m)}$, where E is a polynomial.

Now we have all the tools needed to get an adequate picture of the recurrence relation corresponding to equation (2.6). We use the Ansatz $N(y) = \sum_{i=0}^{\infty} \epsilon_{i+1} y^i$ in equation (3.8).

The constant term. The first step in solving our recurrence relation is to get an equation for ϵ_1 .

Lemma 10. *Let $N(y) = \sum_{i=0}^{\infty} \epsilon_{i+1} y^i$. The values of λ for which $\lambda^{-k} S_\lambda(yN)$ has no constant term are precisely the solutions of $R(\epsilon_1, \lambda) = 0$. Here $R(\epsilon_1, \lambda)$ is given by*

$$R(\epsilon_1, \lambda) := [\lambda^{-k} S_\lambda(yN)]_0 = \sum_{i=0}^k a_{ii} \epsilon_1^i + \lambda^{-1} E(\epsilon_1, \lambda^{-1}) \quad (3.10)$$

for some $E(\epsilon_1, \lambda^{-1})$ which is a polynomial of degree at most k in ϵ_1 (and it has degree at most $k-2$ in λ^{-1}).

Proof. Part 1 of Lemma 9 and part 3 of Lemma 8 imply that $[P_i(y)N^i]_0 = a_{ii}\epsilon_1^i$ and $[P_i(y)N^{i-1-j}y^j\lambda^{-j}N^{(j)}]_0 = 0$ for all $j \geq 1$. Hence by Lemma 7 the constant term of $\lambda^{-k}S_\lambda(yN)$ is given by

$$\sum_i a_{ii}\epsilon_1^i + [b]_0.$$

By Lemma 9 part 3, there is a polynomial E such that $[b]_0 = \lambda^{-1}E(\epsilon_1, \lambda^{-1})$. Part 1 of this same lemma gives that the only terms in b contributing a non-zero term of degree j in ϵ_1 to $[b]_0$ come from terms of the form $c(\lambda^{-1})^{\alpha_1}N^j$ with $\alpha_1 \geq 1$ and $c \in \mathbb{C}$. It is clear that such a term will have $j \leq k$ since by (3.7) and (3.1) it stems from some $\lambda^{-i}y^{-i}\Theta(R_i) = y^{-i}(yN - y^2\lambda^{-1}D_y)^i * 1$ with $i \leq k$, which contains N at most in the power i . By the same observation it follows that the degree in λ^{-1} of such a term is less than or equal to $k - 1$. \square

Hence the initial step in computing the formal solution to (3.7) is solving the equation

$$R(\epsilon_1, \lambda) = \sum_{i=0}^k a_{ii}\epsilon_1^i + \lambda^{-1}E(\epsilon_1, \lambda^{-1}) = 0. \quad (3.11)$$

Note that this equation tends to the equation $\sum_{i=0}^k a_{ii}\epsilon_1^i = 0$ as $\lambda \rightarrow \infty$ and that the latter coincides with equation (2.7). In particular, under the assumptions of Theorem 1, or equivalently, of Lemma 2, equation (3.11) has k distinct solutions for any sufficiently large value of λ . Choose one of the branches $\epsilon_1 = \epsilon_1(\lambda)$ that solves (3.11) in a neighborhood of $\lambda = \infty$ in \mathbb{CP}^1 . This means that we have determined ϵ_1 for any large enough value of λ and we can start determining the other coefficients of $N(y) = \sum_{i=0}^\infty \epsilon_{i+1}y^i$ in terms of the chosen ϵ_1 .

The recurrence formula. Assume now that the coefficients $\epsilon_1, \dots, \epsilon_m$ of N have already been defined. The recursive step that defines ϵ_{m+1} then corresponds to solving the equation $[\lambda^{-k}S_\lambda(yN)]_m = 0$.

Lemma 11. *One has*

$$[\lambda^{-k}S_\lambda(yN(y))]_m = \Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1}m)\epsilon_{m+1} + B(\epsilon_1, \dots, \epsilon_m, \lambda^{-1}, \lambda^{-1}m),$$

where B and Φ_0 are polynomials and the latter satisfies

$$\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1}m) = -\sum_{i=0}^k a_{ii} \frac{(\epsilon_1 - \lambda^{-1}m)^i - \epsilon_1^i}{\lambda^{-1}m} + \lambda^{-1}E(\epsilon_1, \lambda^{-1}, \lambda^{-1}m), \quad (3.12)$$

where E is a polynomial whose degree in the third variable (which is $\lambda^{-1}m$) is strictly less than $k-2$. The degree of B in the last variable (which is $\lambda^{-1}m$ as well) is less than or equal to $k-1$.

Proof. By equation (3.9) one has

$$\begin{aligned} [\lambda^{-k}S_\lambda(yN)]_m &= \sum_i [P_iN^i]_m + \\ &\quad \sum_{i=0}^k \sum_{j=1}^{i-1} \binom{i}{j+1} (-1)^j [P_iN^{i-1-j}y^j\lambda^{-j}N^{(j)}]_m + [b]_m. \end{aligned} \quad (3.13)$$

By Lemma 8 the following two relations hold $\text{mod}(\epsilon_1, \dots, \epsilon_m)$:

$$[P_iN^i]_m = a_{ii}i\epsilon_1^{i-1}\epsilon_{m+1} = a_{ii} \binom{i}{1} \epsilon_1^{i-1}\epsilon_{m+1}$$

and

$$[P_iN^{i-1-j}y^j\lambda^{-j}N^{(j)}]_m = a_{ii}\lambda^{-j}m(m-1)\dots(m-j+1)\epsilon_1^{i-1-j}\epsilon_{m+1}.$$

Finally, by Lemma 9 one has $[b]_m = \lambda^{-1}E_2(\epsilon_1, \lambda^{-1}, \lambda^{-1}m)\epsilon_{m+1}$, where E_2 is a polynomial. The identity $\lambda^{-j}m(m-1)\dots(m-j+1) = \lambda^{-j}m^j + \lambda^{-1}F_i(\lambda^{-1}, \lambda^{-1}m)$, where F_i is a polynomial, implies that $\text{mod}(\epsilon_1, \dots, \epsilon_m)$ one further has

$$[\lambda^{-k}S_\lambda(yN)]_m = \left(\sum_{i=0}^k a_{ii} \sum_{j=0}^{i-1} \binom{i}{j+1}\right) (-1)^j \epsilon_1^{i-1-j} \lambda^{-j} m^j + \lambda^{-1}E(\epsilon_1, \lambda^{-1}, \lambda^{-1}m)\epsilon_{m+1}.$$

The sum over j in the latter expression is $-\lambda m^{-1}((\epsilon_1 - \lambda^{-1}m)^i - \epsilon_1^i)$. This completes the proof of the formula for Φ_0 . The bounds for the degrees of the involved polynomials stated in the lemma follow directly from the last statement of Lemma 7 and part (5) of Lemma 8. \square

Remark 5. Once we prove – which we will do in the next section – that there is an open set D containing $z = \infty$ where $\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1}m) \neq 0$ for all m and $\lambda \in \Omega$ then by Lemma 11 we immediately get a recursive determination of $\epsilon_{m+1}(\lambda)$. Indeed, the initial data is the choice of $\epsilon_1(\lambda)$. In (3.11) we saw that $\lim_{\lambda \rightarrow \infty} \epsilon_1(\lambda)$ is a root of (2.7) so in particular it is bounded. Since in the right-hand side of (3.13) the only part not being a multiple of λ^{-1} is $\sum_i [P_i N^i]_m$, we get that $yN_\lambda(y) = \sum_{i=1}^\infty \epsilon_i(\lambda)y^i$ converges formally to a formal power series $yN(y) = \sum_{i=1}^\infty \epsilon_i y^i$ which is a formal solution to the equation $\sum_i P_i N^i = 0$. This shows that $yN(y)$ is a formal solution (at infinity) to the equation of the plane curve associated with the pencil T_λ .

Summary. We have considered the Ansatz

$$L(z) = yN(y) = \sum_{i=1}^\infty \epsilon_i y^i$$

for equation (2.6) at $z = \infty$ in the form given by the change of variable $y = 1/z$ (cf. (3.7)). We saw by Lemma 10 that $\epsilon_1 = \epsilon_1(\lambda)$ has to be a solution to equation (3.11) and that it is always possible to find such a solution for large λ under the assumption of Theorem 1. Furthermore, Lemma 11 is immediately reformulated into the recurrence relation

$$\epsilon_{m+1} = -\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1}m)^{-1} B(\epsilon_1, \dots, \epsilon_m, \lambda^{-1}, \lambda^{-1}m), \quad (3.14)$$

which formally solves equation (3.7) under the assumption that for all $m = 1, 2, \dots$ one has $\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1}m) \neq 0$. Clearly, the next step is to study this last condition. This will be done in the next section.

Example 1. Consider $T_\lambda = \sum_{i=0}^2 Q_i(z) \lambda^{2-i} \frac{d^i}{dz^i}$ with $Q_i(z) = \sum_{j=0}^i a_{ij} z^j$, $i = 0, 1, 2$. Then

$$\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1}m) = a_{22}(2\epsilon_1 - \lambda^{-1}m) + a_{11} - a_{22}\lambda^{-1}$$

with $E = -a_{22}$.

The tricky point in proving that there actually exists a formal solution to (2.6) is that the polynomials $\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1}m)$ might in fact vanish for some values of m . Moreover, proving the convergence of formal solutions is also rendered difficult by the possibility that $\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1}m) \rightarrow 0$ when $m \rightarrow \infty$. We can see from Lemma 11 and the above example that the terms in $\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1}m)$ have degree in m that is less than or equal to their total degree in λ^{-1} . This observation will be the crucial point in the proof of the convergence of formal power series solutions given in the next section.

4. ESTIMATING THE RADIUS OF CONVERGENCE

We use the classical method of majorants (see e.g. [18] or [19]) to show that under certain conditions the formal power series solutions to equation (3.7) actually represent analytic functions. The idea is to find an upper bound for the solutions to the studied recurrence relation in terms of a simpler and explicitly solvable one. In other words, we want to substitute the original recurrence relation by a simpler recurrence ensuring that during this process the absolute value of the solutions will not decrease. The recurrence relation (3.14) has, for fixed λ , the form

$$\epsilon_{m+1} = G_m(\epsilon_1, \dots, \epsilon_m), \quad (4.1)$$

where $G_m(\epsilon_1, \dots, \epsilon_m) = \sum_I \delta_I^m \epsilon^I$, $m = 1, 2, \dots$, is a family of polynomials. The following lemma is borrowed from [18].

Lemma 12. *Given a recurrence relation (4.1) assume that*

$$H_m(\epsilon_1, \dots, \epsilon_m) = \sum_I A_I \epsilon^I, \quad m = 1, 2, \dots,$$

is a family of polynomials with positive coefficients satisfying

$$|G_m(\epsilon_1, \dots, \epsilon_m)| \leq H_m(|\epsilon_1|, \dots, |\epsilon_m|) \quad (4.2)$$

for all $\epsilon_i \in \mathbb{C}$. (This is true, for example, if $A_I^m \geq |\delta_I^m|$). Take $E_1 \geq |\epsilon_1|$ and define inductively $E_{m+1} = H_m(E_1, \dots, E_m)$. Then for any m one has that $E_m \geq |\epsilon_m|$ and hence the radius of convergence of the series $\sum_{i=1}^{\infty} \epsilon_i t^i$ is greater than or equal to the radius of convergence of the series $\sum_{i=1}^{\infty} E_i t^i$.

Lemma 12 implies the following statement.

Proposition 4. *Consider a set $\Lambda \subset \mathbb{CP}^1$ containing ∞ and such that $\lambda = 0$ is an interior point in $\mathbb{CP}^1 \setminus \Lambda$. Let $\epsilon_1 = \epsilon_1(\lambda)$, $\lambda \in \Lambda$, be a continuous solution to (3.11). If the polynomials $\Phi_0(\epsilon_1(\lambda), \lambda^{-1}, \lambda^{-1}m)$, $m = 1, 2, \dots$, satisfy the condition*

$$\sup\{|\Phi_0(\epsilon_1(\lambda), \lambda^{-1}, \lambda^{-1}m)^{-1}(\lambda^{-1}m)^r| : \lambda \in \Lambda, m \geq 0, r = 0, 1, \dots, k-1\} < \infty \quad (4.3)$$

then there exists $D > 0$ such that the series $\sum_{i \geq 1} \epsilon_i y^i$ defined by the recurrence (3.14) converges for any $\lambda \in \Lambda$ in the disk $|y| < D$.

Proof. By Lemma 7 we know that

$$\lambda^{-k} S_\lambda(yN) = \sum c_{\alpha_1 \alpha_2 J} (\lambda^{-1})^{\alpha_1} y^{\alpha_2} (\lambda^{-1}y)^{|J|} N^J, \quad (4.4)$$

where the sum is taken over a finite set of indices $\alpha_i \in \mathbb{Z}_{\geq 0}$ and $J \in FS$ such that $\alpha_1 + |J| \leq k-1$.

The choice of ϵ_1 (depending on λ) determines $N(y) = \epsilon_1 + M(y)$, where $M(y) = \sum_{i=1}^{\infty} \epsilon_{i+1} y^i$ has no constant term. If J starts with precisely l zeros, i.e., $J = 0(l) \cup J_+$, then

$$N^J = (\epsilon_1 + M)^l M^{J_+} = \sum_{s=0}^l \binom{l}{s} \epsilon_1^{l-s} M^{0(s) \cup J_+} = \sum_{J_1 \in W} d_{J_1} M^{J_1},$$

where all J_1 's that occur in the last sum satisfy the relation $|J_1| = |J|$. Using this and substituting $M(y)$ into (4.4), we get the equation

$$\sum d_{\alpha_1 \alpha_2 I} (\lambda^{-1})^{\alpha_1} y^{\alpha_2} (\lambda^{-1}y)^{|I|} M^I = 0, \quad (4.5)$$

where the relation $\alpha_1 + |I| \leq k-1$ is still valid. Note that (4.5) contains no non-zero $d_{\alpha_1 \alpha_2 J}$ such that $\alpha_2 = 0$ and $|J| = 0$ since this case will contribute to a nontrivial constant term (degree 0 in y). But the constant term was already eliminated by

the explicit choice of ϵ_1 . The coefficients in (4.5) are polynomials in $\epsilon_1(\lambda)$. Since by the discussion of the asymptotic behavior of equation (3.11) for $\epsilon_1(\lambda)$ we know that $\lim_{\lambda \rightarrow \infty} \epsilon_1(\lambda)$ is finite, these coefficients are bounded in the domain Λ .

Now we will analyze how the recurrence formula (3.14) is affected by our choice of a branch $\epsilon_1(\lambda)$. We will do this in a way similar to the arguments in Lemma 11. Since the constant term of M vanishes by Lemma 8 parts (2)-(4) one gets that only the terms $(\lambda^{-1})^{\alpha_1} y^{\alpha_2} (\lambda^{-1} y)^{|I|} M^I$ for which $I = (l)$ has length 1 and $\alpha_2 = 0$ will satisfy the condition that the m -th coefficient is non-zero mod $(\epsilon_1, \dots, \epsilon_m)$. For these terms one has

$$[(\lambda^{-1})^{\alpha_1} (\lambda^{-1} y)^l M^{(l)}]_m = (\lambda^{-1})^{\alpha_1} m(m-1) \dots (m-l+1) \epsilon_{m+1}.$$

Hence we can split the terms in the left-hand side of equation (4.5) into two groups, namely, the sum $\sum_{\alpha_1, l} d_{\alpha_1 0(l)} (\lambda^{-1})^{\alpha_1} (\lambda^{-1} y)^l M^{(l)}$ and the remaining terms $\sum_V d_{\alpha_1 \alpha_2 I} (\lambda^{-1})^{\alpha_1} y^{\alpha_2} (\lambda^{-1} y)^{|I|} M^I$. These remaining terms correspond to a certain index set V . The first sum has the m -th coefficient equal to $\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1} m) \epsilon_{m+1}$, while the m -th coefficient in the second sum is a polynomial in the preceding coefficients. By the hypothesis of Proposition 4 we have $\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1} m) \neq 0$ if $\lambda \in \Lambda$. Hence the recurrence relation expressing ϵ_{m+1} can be rewritten as

$$\epsilon_{m+1} = -\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1} m)^{-1} \sum_V d_{\alpha_1 \alpha_2 I} [(\lambda^{-1})^{\alpha_1} y^{\alpha_2} (\lambda^{-1} y)^{|I|} M^I]_m. \quad (4.6)$$

Interpret its right-hand side as a polynomial $G_m(\epsilon_2, \dots, \epsilon_m)$ with coefficients depending on λ . If $K = \max\{|d_{\alpha_1 \alpha_2 I}|, (\alpha_1, \alpha_2, I) \in V\}$ then clearly

$$|G_m(\epsilon_2, \dots, \epsilon_m)| \leq K \sum_V |\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1} m)^{-1} [(\lambda^{-1})^{\alpha_1} y^{\alpha_2} (\lambda^{-1} y)^{|I|} M^I]_m|. \quad (4.7)$$

We can find an even simpler upper bound for these terms. Set $J = (j_1, j_2, \dots, j_s)$ and recall that

$$[y^{|J|} M^J]_m = \sum_{c_1 + \dots + c_s = m} \prod_{i=1}^s c_i(c_i - 1) \dots (c_i - j_i + 1) \epsilon_{c_i}$$

by Lemma 8. Clearly, $\prod_{i=1}^s c_i(c_i - 1) \dots (c_i - j_i + 1) \leq m^{|J|}$ and thus we get

$$|[y^{|J|} M^J]_m| \leq m^{|J|} \sum_{c_1 + \dots + c_s = m} \prod_{i=1}^s |\epsilon_{c_i}|.$$

Setting $M_{abs} := \sum_{i=2}^{\infty} |\epsilon_i| t^i$ and noting that $s = \text{lng}(J)$ we obtain

$$\sum_{c_1 + \dots + c_s = m} \prod_{i=1}^s |\epsilon_{c_i}| = [M_{abs}^{\text{lng}(J)}]_m$$

and so we have the bound

$$|[y^{|J|} M^J]_m| \leq m^{|J|} [M_{abs}^{\text{lng}(J)}]_m.$$

This implies the inequality

$$|G_m(\epsilon_2, \dots, \epsilon_m)| \leq K \sum_V |\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1} m)^{-1} \lambda^{-\alpha_1} (\lambda^{-1} m)^{|I|} [y^{\alpha_2} M_{abs}^{\text{lng}(I)}]_m|. \quad (4.8)$$

As it was mentioned in the beginning of the proof, for each term in this sum one has $\alpha_1 + |I| \leq k - 1$. Hence both inequalities $\alpha_1 \leq k - 1$ and $|I| \leq k - 1$ hold. The assumption that 0 is an interior point of the complement of Λ implies that there is a positive real number D such that $|\lambda^{-i}| \leq D$ for $\lambda \in \Lambda$ and $i = 1, \dots, k - 1$. By the second assumption of the proposition there is a constant C such that

$$|\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1} m)^{-1} (\lambda^{-1} m)^{|I|}| \leq C$$

for $\lambda \in \Lambda$ and any I occurring in (4.8). So

$$|G_m(\epsilon_2, \dots, \epsilon_m)| \leq KCD \sum_V [y^{\alpha_2} M_{abs}^{lng(I)}]_m \leq L \sum_{i,j} [y^i M_{abs}^j]_m =: H_m(|\epsilon_2|, \dots, |\epsilon_m|)$$

for some positive real number $L := KCDr$, where r is the maximal number of indices in V (the index set introduced above) that correspond to some $i = \alpha_2$ and $j = lng(I)$. Note that L is independent of m , since the index set V just depends on k . The summation in H_m is taken over all pairs of non-negative integers (i, j) except $(0, 0)$ and $(0, 1)$. (The fact that $(0, 0)$ is not needed follows from the observation that there are no terms in (4.5) with $\alpha_2 = 0$ and $lng(I) = 0$; the fact that $(0, 1)$ is not needed is implied by the observation that the terms with $\alpha_2 = 0$ and $lng(I) = 1$ were excluded from V by the splitting of terms in (4.5) for the construction of V .) This is an infinite sum, but we still get a well-defined polynomial since M contains no constant term.

The recurrence formulae $\epsilon_{m+1} = H_m(\epsilon_2, \dots, \epsilon_m)$, $m \geq 1$, correspond to equating the m -th coefficients in both sides of the relation

$$M = L \sum_{i,j} y^i M^j = L \left(\frac{1}{(1-y)(1-M)} - 1 - M \right)$$

defining the power series $M(y) = \sum_{i=1}^{\infty} \epsilon_{i+1} y^i$. (Note that for $m = 1$ we simply get $\epsilon_2 = H_1 \in \mathbb{R}$.) The above equation is equivalent to the quadratic equation

$$(1+L)M^2 - M + Ly/(1-y) = AM^2 - M + C = 0$$

and its solutions are $\frac{1}{2A} \pm \frac{\sqrt{1-4CA}}{2A}$. These solutions are univalent analytic functions in case $|4CA| < 1$. The latter condition is easily seen to be satisfied for $|y| < \frac{1}{(1+2L)^2}$. These solutions are developable in power series in this disk and their power series will satisfy our recurrence formulae. This accomplishes the proof of Proposition 4. Indeed, we have found a sequence of polynomials that satisfy the assumptions of Lemma 12 and proved that the formal power series that they define is analytically convergent in some disk around $y = 0$. (This argument mimics the elegant proof that an algebraic function has a convergent power series expansion given in [18, pp. 34-35].) \square

Remark 6. It is worth mentioning that if we know the supremum in Proposition 4 we can easily get the estimate $\frac{1}{(1+2L)^2}$ for the radius of convergence of the solutions to equation (3.9) at least for small k .

Convergence of polynomial solutions. We will now show that the condition on the supremum in Proposition 4 is generically satisfied for the families of polynomial eigenfunctions used in Propositions 1 and 3. This means that we assume that the corresponding pencil is of general type. If $p_n(z)$ is any monic degree n polynomial then its normalized logarithmic derivative $L_n(z) = \frac{p'_n(z)}{\lambda_n p_n(z)}$ expanded at infinity using the variable $y = 1/z$ satisfies the relation

$$L_n(z) = yN(y) = \frac{n}{\lambda_n} y + \dots = \sum_{i=1}^{\infty} \epsilon_{i,n} y^i.$$

In particular, $\epsilon_{1,n} = \frac{n}{\lambda_n}$. (Note that so far $\lambda_n \neq 0$ is any constant.)

Now fix λ_n to be one of the families of eigenvalues in Propositions 1. In a certain neighborhood of infinity Λ there is then for n so large that $\lambda_n \in \Lambda$ a unique degree

n eigenpolynomial $p_n(z)$ that satisfies $T_{\lambda_n} p_n(z) = 0$. By Proposition 1 for n large enough one has $\lambda_n \sim \alpha n$, where α is one of the k distinct roots of (1.3). This gives

$$\lambda_n^{-1} n = \epsilon_{1,n} \rightarrow 1/\alpha$$

as $n \rightarrow \infty$. Note that $1/\alpha$ is a root of the reciprocal characteristic equation (2.7):

$$a_{k,k} \xi^k + a_{k-1,k-1} \xi^{k-1} + \dots + a_{0,0} = 0. \quad (4.9)$$

We next show that the assumption in Proposition 4 ensuring the convergence of our formal power series solutions is satisfied for a large class of cases.

Proposition 5. *Assume that*

- (i) *the polynomial $\sum_{i=0}^k a_{ii} t^i$ has degree k (i.e., $a_{kk} \neq 0$)*
- (ii) *the root α^{-1} of $\sum_{i=0}^k a_{ii} t^i = 0$ is simple,*
- (iii) *for any $r > 0$ the number $(1-r)\alpha^{-1}$ is never a root of this same polynomial (in particular, taking $r = 1$ implies that 0 is not a root of the above polynomial, which gives $a_{00} \neq 0$).*

Then there exists a neighborhood $\tilde{\Lambda} \in \mathbb{C}$ of infinity with the following property: if $n \in \mathbb{Z}_{\geq 0}$ is such that $\lambda_n \in \tilde{\Lambda}$ and $\epsilon_{1,n} = \frac{n}{\lambda_n}$, then

$$\sup\{|\Phi_0(\epsilon_{1,n}, \lambda_n^{-1}, \lambda_n^{-1} m)^{-1} (\lambda_n^{-1} m)^r| : m \geq 0, r = 0, 1, \dots, k-1\} < \infty$$

and so the condition in Proposition 4 is valid.

Proof. Recall first that by Lemma 11 we have

$$\Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1} m) = - \sum_{l=0}^k a_{ll} \frac{(\epsilon_1 - \lambda^{-1} m)^l - \epsilon_1^l}{\lambda^{-1} m} + \lambda^{-1} E(\epsilon_1, \lambda^{-1}, \lambda^{-1} m). \quad (4.10)$$

We need to establish the existence of a finite supremum in the previous proposition. Argue by contradiction: if the proposition is not true there are sequences of values m_j and n_j , $j = 1, 2, \dots$, such that

- $\lim_{j \rightarrow \infty} n_j = \infty$,
- for some $i \leq k-1$ one has $(\lambda_{n_j}^{-1} m_j)^{-i} \Phi_0(\epsilon_{1,n_j}, \lambda_{n_j}^{-1}, \lambda_{n_j}^{-1} m_j) \rightarrow 0$ as $j \rightarrow \infty$.

We will in the following use the notation $n = n_j$, $m = m_j$, $\epsilon_1 = \epsilon_{1,n} = \frac{n}{\lambda_n}$, and $\lambda = \lambda_n \in \Lambda$, where $n \in \mathbb{Z}_{\geq 0}$, and thus suppress indices in the sequence. All limits will be taken as $j \rightarrow \infty$.

By taking a subsequence we see that there exist two possibilities: either there is a sequence as above for which m/n converges to a finite limit $r \in \mathbb{R}$, or there is such a sequence for which $m/n \rightarrow \infty$. In the second case we first note that also $\lambda^{-1} m \rightarrow \infty$ (since $\lim \lambda^{-1} m = \lim \lambda^{-1} n \cdot \lim(m/n) = \alpha^{-1} \lim m/n$). Hence by our assumption also $(\lambda^{-1} m)^{-(k-1)} \Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1} m) \rightarrow 0$. To get a contradiction we use the fact that the degree in $\lambda^{-1} m$ of E is less than $k-2$ by Lemma 11, and that $\lambda^{-1} \rightarrow 0$ for the chosen sequence (recall that λ depends on n_j and $n_j \rightarrow \infty$). It follows that $\lim(\lambda^{-1} m)^{-(k-1)} E(\epsilon_1, \lambda^{-1}, \lambda^{-1} m) = 0$. Then the hypothesis that $a_{kk} \neq 0$ and the explicit description in (4.10) imply that $\lim(\lambda^{-1} m)^{-(k-1)} \Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1} m) = (-1)^{k-1} a_{kk} \neq 0$, which is the desired contradiction.

In the first case, the hypothesis implies that (for our sequence of m and n) $\lambda^{-1} m = (\lambda^{-1} n)(m/n) \rightarrow r/\alpha$. Furthermore

$$\lim \Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1} m) = (r/\alpha)^i \lim((\lambda^{-1} m)^{-i} \Phi_0(\epsilon_1, \lambda^{-1}, \lambda^{-1} m)) = 0. \quad (4.11)$$

Now we also have $\lambda \rightarrow \infty$, which implies that $\lim \lambda^{-1} E(\epsilon_1, \lambda^{-1}, \lambda^{-1} m) = 0$ and thus (4.11) yields

$$\lim \sum_{l=0}^k a_{ll} \frac{(\epsilon_1 - \lambda^{-1} m)^l - \epsilon_1^l}{\lambda^{-1} m} = 0. \quad (4.12)$$

If $r \neq 0$ this immediately implies that both $\alpha^{-1} - r\alpha^{-1}$ and $\alpha^{-1} = \lim \epsilon_1$ solve the same equation, namely $\sum_{l=0}^k a_l t^l = 0$. This gives a contradiction to the assumed conditions on this equation. On the other hand, if $r = 0$ then a calculation shows that the left-hand side of (4.12) is $\frac{d}{dt}(\sum_{l=0}^k a_l t^l)|_{t=\alpha^{-1}}$, which contradicts the fact that α^{-1} is a simple root and finishes the proof. \square

5. PROOF OF THEOREMS 1 AND 2

We start with Theorem 2. Consider a pencil T_λ of general type as defined in the introduction. By Proposition 1 its polynomial eigenfunctions split into k distinct families $p_{n,j}$, $j = 1, \dots, k$. Fix one of these families $p_n := p_{n,j}$ with corresponding eigenvalues $\lambda_n \sim \alpha n$ as $n \rightarrow \infty$.

For each λ we have an equation (2.6). Among its solutions in an open set Ω are all normalized logarithmic derivatives $\frac{p'}{\lambda p}$ of eigenfunctions p (i.e., $T_\lambda * p = 0$) that are defined and non-vanishing in Ω . The recurrence relation (3.14) – that we studied in the two last sections – solves equation (2.6) by means of the power series expansion at $y^{-1} = z = \infty$ of a solution of the special form $yN(y) = \sum_{i=0}^{\infty} \epsilon_{i+1} y^{i+1}$ (see Lemma 11 in §3). In Lemma 10 it was shown that for each λ large enough there are k possible initial values of $\epsilon_1 = \epsilon_1(\lambda)$. Let $\lambda = \lambda_n$ be large enough for this to be true. Since T_λ is of general type the condition in Proposition 5 is satisfied. Hence by Proposition 4 there exists a non-trivial disk D around ∞ where for any n all k formal solutions to the recurrence relation (3.14) converge to analytic functions.

Consider now $L_n(z) = \frac{p'_n(z)}{\lambda_n p_n(z)}$ for an arbitrarily fixed n . This is a solution to (2.6) and has the required zero at infinity, so there is a neighborhood of ∞ where $L_n(z)$ can be expanded as a convergent power series $yN_n(y)$. In particular, this power series will be one of the k solutions of the recurrence relation and thus it will actually converge in D . It follows that the zeros of the polynomials p_n are contained in the compact region of the complex plane that is the complement of D . This fact immediately implies Theorem 2.

Let us now concentrate on Theorem 1 and start by proving the last part of this theorem. Recall that by Remark 5 the recurrence formulas (3.14) converge as $\lambda \rightarrow \infty$ to recursive formulas determining formal solutions to the algebraic equation (1.2). This implies that the formal power series at ∞ for $L_n(z)$ converges formally (i.e., coefficientwise) to the formal power series at ∞ of the algebraic function $\gamma(z)$ for which $\lim_{z \rightarrow \infty} z\gamma(z) = \alpha^{-1}$ (see Lemma 2 in §2). Since the convergence radius has a uniform non-zero lower bound, this gives that the convergence is uniform in a neighborhood of ∞ . The different branches of the algebraic function $\gamma(z)$ at ∞ correspond to the different values of α and this makes it clear that they all occur (cf. Lemma 2). This proves the last part of Theorem 1.

To settle the first part of Theorem 1 we need some basic properties of probability measures. If K is a compact set in \mathbb{C} denote by $M(K)$ the space of all probability measures supported in K equipped with the weak topology. It is known that $M(K)$ is a sequentially compact Hausdorff space, which allows one to choose a convergent subsequence from any sequence of measures belonging to $M(K)$ (which is the statement of Helly's theorem).

Lemma 13 (cf. Lemma 8 of [4]). *Let $\{q_m(z)\}$ be a sequence of polynomials with $\deg q_m(z) \rightarrow \infty$ as $m \rightarrow \infty$. Denote by μ_m and μ'_m the root-counting measures of $q_m(z)$ and $q'_m(z)$, respectively, and assume that there exists a compact set K containing the supports of all measures μ_m and therefore also the supports of all measures μ'_m . If $\mu_m \rightarrow \mu$ and $\mu'_m \rightarrow \mu'$ as $m \rightarrow \infty$ and u and u' are the logarithmic potentials of μ and μ' , respectively, then $u' \leq u$ in \mathbb{C} with equality in the unbounded component of $\mathbb{C} \setminus \text{supp}(\mu)$.*

Example 2. Consider the polynomial sequence $\{z^m - 1\}$. The measure μ is then the uniform distribution on the unit circle of total mass 1. Its logarithmic potential $u(z)$ equals $\log |z|$ if $|z| \geq 1$ and 0 in the disk $|z| \leq 1$. On the other hand, the sequence of derivatives is given by $\{mz^{m-1}\}$ and the corresponding (limiting) logarithmic potential $u'(z)$ equals $\log |z|$ in $\mathbb{C} \setminus \{0\}$. Obviously, $u(z) = u'(z)$ in $|z| \geq 1$ and $u'(z) < u(z)$ in $|z| < 1$.

In the notation of Theorem 1 consider the family of eigenpolynomials $\{p_{n,j}(z)\}$ for some arbitrarily fixed value of the index $j = 1, \dots, k$. Assume that N_j is a subsequence of the natural numbers such that

$$\mu_j^{(i)} := \lim_{n \rightarrow \infty, n \in N_j} \mu_{n,j}^{(i)} \quad (5.1)$$

exists for $i = 0, \dots, k$, where $\mu_{n,j}^{(i)}$ denotes the root-counting measure of $p_{n,j}^{(i)}(z)$. The existence of such N_j follows from the above remark on $M(K)$ (Helly's theorem). Notice that for each i the logarithmic potential $u_j^{(i)}$ of $\mu_j^{(i)}$ satisfies a.e. the identity

$$u_j^{(i)}(z) - u_j^{(0)}(z) = \lim_{n \rightarrow \infty, n \in N_j} \frac{1}{n} \log \left| \frac{p_{n,j}^{(i)}(z)}{n(n-1) \dots (n-i+1)p_{n,j}(z)} \right|.$$

The next proposition completes the proof of Theorem 1 and also shows the remarkable property that if one considers a sequence of eigenpolynomials for some spectral pencil then the situation $u'(z) < u(z)$ seen in Example 2 can never occur.

Proposition 6. The measures $\mu_j^{(i)}$, $i = 0, \dots, k$, are all equal and the scalar multiple $\tilde{\Psi}_j = C_\mu/\alpha_j$ of the Cauchy transform of this common measure μ_j satisfies equation (1.2) almost everywhere.

Proof. For $n \in N_j$ one has

$$\frac{p_{n,j}^{(i+1)}(z)}{(n-i)p_{n,j}^{(i)}(z)} \rightarrow \int_{\mathbb{C}} \frac{d\mu_j^{(i)}(\zeta)}{z-\zeta} \text{ as } n \rightarrow \infty$$

with convergence in L_{loc}^1 . The well-known property of convergence in L_{loc}^1 implies that passing to a subsequence one can assume that the above convergence is actually the pointwise convergence almost everywhere in \mathbb{C} . Recall that $p_{n,j}(z)$ satisfies the differential equation $T_{\lambda_{n,j}} p_{n,j}(z) = 0$, i.e.,

$$Q_k(z)p_{n,j}^{(k)}(z) + \lambda_{n,j}Q_{k-1}(z)p_{n,j}^{(k-1)}(z) + \dots + \lambda_{n,j}^k Q_0(z)p_{n,j}(z) = 0. \quad (5.2)$$

Therefore,

$$Q_k(z) \frac{p_{n,j}^{(k)}(z)}{p_{n,j}(z)} = -\lambda_{n,j}Q_{k-1}(z) \frac{p_{n,j}^{(k-1)}(z)}{p_{n,j}(z)} - \dots - \lambda_{n,j}^k Q_0(z).$$

The latter relation implies that $\lim_{n \rightarrow \infty, n \in N_j} \frac{p_{n,j}^{(k)}(z)}{n(n-1) \dots (n-k+1)p_{n,j}(z)}$ exists almost everywhere in \mathbb{C} . Indeed,

$$\begin{aligned} \frac{p_{n,j}^{(k)}(z)}{p_{n,j}(z)} &= -\frac{1}{Q_k(z)} \sum_{l=1}^k \lambda_{n,j}^l Q_{k-l}(z) \frac{p_{n,j}^{(k-l)}(z)}{p_{n,j}(z)} \\ &= -\frac{1}{Q_k(z)} \left(\prod_{i=1}^{k-1} (n-k+1+i) \cdot \lambda_{n,j} Q_{k-1}(z) \cdot \prod_{i=1}^{k-1} \frac{p_{n,j}^{(k-i)}(z)}{(n-k+1+i)p_{n,j}^{(k-1-i)}(z)} \right. \\ &\quad \left. + \prod_{i=1}^{k-2} (n-k+2+i) \cdot \lambda_{n,j}^2 Q_{k-2}(z) \cdot \prod_{i=1}^{k-2} \frac{p_{n,j}^{(k-1-i)}(z)}{(n-k+2+i)p_{n,j}^{(k-2-i)}(z)} \right) \end{aligned}$$

$$+ \dots + \lambda_{n,j}^k Q_0(z) \frac{p_{n,j}^{(k)}(z)}{p_{n,j}(z)} \Bigg).$$

Thus

$$\begin{aligned} \frac{p_{n,j}^{(k)}(z)}{\prod_{i=1}^k (n-k+i) \cdot p_{n,j}(z)} &= -\frac{1}{Q_k(z)} \left(\frac{\lambda_{n,j} Q_{k-1}(z)}{(n-k+1)} \prod_{i=1}^{k-1} \frac{p_{n,j}^{(k-i)}(z)}{(n-k+1+i) p_{n,j}^{(k-1-i)}(z)} \right. \\ &\quad \left. + \frac{\lambda_{n,j}^2 Q_{k-2}(z)}{(n-k+1)(n-k+2)} \prod_{i=1}^{k-2} \frac{p_{n,j}^{(k-1-i)}(z)}{(n-k+2+i) p_{n,j}^{(k-2-i)}(z)} + \dots + \frac{\lambda_{n,j}^k Q_0(z)}{\prod_{i=1}^k (n-k+i)} \right). \end{aligned}$$

Using the asymptotics $\lambda_{n,j} \sim \alpha_j n$ and the pointwise convergence a.e. mentioned in the beginning of the proof we get that the right-hand side of the latter expression is convergent a.e. in \mathbb{C} . Therefore,

$$u_j^{(k)}(z) - u_j^{(0)}(z) = \lim_{n \rightarrow \infty, n \in N_j} \frac{1}{n} \log \left| \frac{p_{n,j}^{(k)}(z)}{n(n-1) \dots (n-k+1) p_{n,j}(z)} \right| = 0$$

almost everywhere in \mathbb{C} . On the other hand, $u_j^0 \geq u_j^{(1)} \geq \dots \geq u_j^{(k)}$ by Lemma 13. Hence the potentials $u_j^{(i)}$ are all equal and the corresponding measures $\mu_j^{(i)} = \Delta u_j^{(i)} / 2\pi$ are equal as well.

The fact that the multiple $\tilde{\Psi}_j = C_\mu / \alpha_j$ of the Cauchy transform of this common measure μ_j satisfies equation (1.2) almost everywhere follows by rewriting equation (5.2). It is equivalent to

$$Q_k(z) \frac{p_{n,j}^{(k)}(z)}{\lambda_{n,j}^k p_{n,j}(z)} + Q_{k-1}(z) \frac{p_{n,j}^{(k-1)}(z)}{\lambda_{n,j}^{k-1} p_{n,j}(z)} + \dots + Q_0(z) = 0. \quad (5.3)$$

Using the convergence proved above we get

$$\frac{p_{n,j}^{(l)}(z)}{\lambda_{n,j}^l p_{n,j}(z)} = \prod_{r=0}^{l-1} \frac{p_{n,j}^{(r+1)}(z)}{\lambda_{n,j} p_{n,j}^{(r)}(z)} \rightarrow \frac{C_\mu(z)^l}{\alpha_j^l} \text{ as } n \rightarrow \infty, n \in N_j,$$

for $l = 0, \dots, k$ and so by (5.3) we obtain

$$\sum_{i=0}^k Q_i(z) \left(\frac{C_\mu(z)}{\alpha_j} \right)^i = 0,$$

which is the desired equation. \square

This finishes the proof of Theorem 1. For the sake of completeness let us prove the result on L_{loc}^1 -convergence used in the preceding proof.

Lemma 14. *Suppose that μ_n , $n = 1, 2, \dots$, are probability measures that converge as distributions to the measure μ . Then $z^{-1} * \mu_n$ converges to $z^{-1} * \mu$ in L_{loc}^1 .*

Proof. That $z^{-1} * \mu_n \in L_{\text{loc}}^1$ is a consequence of the fact that $z^{-1} \in L_{\text{loc}}^1$ combined with Fubini's theorem (see [16]). Write $z^{-1} = f_\epsilon + g_\epsilon$, where g_ϵ is a test function with compact support and $\|f_\epsilon\|_1 \leq \epsilon$. Then by Fubini's theorem one has $\|f_\epsilon * \mu_n\|_1 \leq \epsilon$. Since the functions $g_\epsilon * \mu_i$ converge to $g_\epsilon * \mu$ uniformly hence also in L_{loc}^1 the lemma follows. \square

6. THE SUPPORT OF GENERATING MEASURES: PROOF OF THEOREMS 3 AND 4

This section is devoted to proving Theorems 3 and 4, which requires a rather elaborate mixture of ideas from algebraic geometry and complex analytic techniques. The latter are based on the main results of [10].

Definition 1. A function Φ defined in a domain $U \subset \mathbb{C}$ is called *piecewise analytic*, or a *PA-function* for short, if there exists a finite family of pairwise distinct analytic functions $\{A_i(z)\}_{i=1}^r$ in U and an associated family of disjoint open sets $\{M_i\}_{i=1}^r$ with $M_i \subset \Omega := \{z \in U \mid A_k(z) \neq A_l(z), 1 \leq k \neq l \leq r\}$, $1 \leq i \leq r$, such that $\{M_i\}_{i=1}^r$ forms a covering of U up to a set of Lebesgue measure 0 and

$$\Phi(z) = \sum_{i=1}^r A_i(z) \chi_i(z) \text{ in } U,$$

where χ_i is the characteristic function of the set M_i , $1 \leq i \leq r$.

Note that *PA-functions* need not be continuous – this will certainly not be the case in our situation. The following result singles out the properties of the limit $\Phi(z) := C_{\mu_j}(z)$ that we will need, see Theorem 1.

Proposition 7. $\Phi(z)$ is a *PA-function* in any simply-connected open set U . More precisely, we have that $\Phi(z) = \sum_{i=1}^r A_i(z) \chi_i(z)$, where χ_i are the characteristic functions of certain subsets of U (defined in the proof below) and A_1, \dots, A_r are different (analytic) branches of the equation

$$\sum_{i=0}^k Q_i(z) \alpha_j^{-i} w^i = 0$$

in U . Moreover, $\mu_j = \partial\Phi(z)/\partial\bar{z} \geq 0$ as a distribution supported in U .

Proof. This is an immediate corollary of Theorem 1 which implies that $\alpha_j^{-1} C_{\mu_j}(z)$ satisfies an algebraic equation (1.2) almost everywhere in \mathbb{C} . Therefore in any simply-connected U one has that $\prod_{i=1}^k (C_{\mu_j}(z) - A_i(z)) = 0$ almost everywhere. Let M_i (with characteristic function χ_i) be the subset of U where $C_{\mu_j}(z) = A_i(z)$. Then we immediately get that $\Phi(z) := C_{\mu_j}(z) = \sum_{i=1}^r A_i(z) \chi_i(z)$ and the subsets M_i cover U up to a set of Lebesgue measure zero. \square

Using some of the main results of [10] (which was initially motivated by the present project) we now derive several consequences for the support of the measure μ_j and prove Theorem 3. If $\Phi(z)$ is any *PA-function* with $r \geq 3$ we associate to each triple (i, j, k) of distinct indices in $\{1, \dots, r\}$ the following set

$$\Gamma_{i,j,k} = \{z \in U \mid \exists c \in \mathbb{R} \text{ such that } A_i(z) - A_k(z) = c(A_j(z) - A_k(z))\}.$$

It is easy to see that this definition only depends on the set $\{i, j, k\}$. Clearly, if $r \leq 2$ these sets are empty.

Theorem 5 (see Theorem 1 and Corollary 2 of [10]). *In the above notation assume that $\partial\Phi/\partial\bar{z} \geq 0$ as a distribution supported in U and let $p \in U$ be such that any sufficiently small neighborhood N of p intersects each M_i , $1 \leq i \leq r$, in a set of a positive Lebesgue measure. Define $H_i(z) = \Re \left[\int_p^z A_i(w) dw \right]$ and suppose also that*

- (i) $A_i(p) \neq A_j(p)$ for $1 \leq i \neq j \leq r$, i.e., p is a non-singular point of $H_i - H_j$,
- (ii) there is at most one $\Gamma_{i,j,k}$ that contains p .

Then there exists a neighborhood of p where $\partial\Phi/\partial\bar{z}$ is supported in a union of segments of 0-level sets to the functions $H_i - H_j$, $1 \leq i \neq j \leq r$.

By Proposition 7 one can apply Theorem 5 to $C_{\mu_j}(z)$. Let Γ_2 be the set of points where conditions (i) and (ii) in the above theorem fail to hold. In other words, Γ_2 is the set of points p which are either one of the finite number of branch points where $A_i(p) = A_k(p)$ for a pair (i, k) of distinct indices, or satisfy $A_i(p) - A_k(p) \in \mathbb{R}(A_j(p) - A_k(p))$ for at least two distinct triples $\{i, j, k\}$ in $\{1, \dots, r\}$. Notice that the original measure μ_j can be restored from its Cauchy transform by

$$\mu_j = \frac{1}{\pi} \frac{\partial C_{\mu_j}(z)}{\partial \bar{z}},$$

where $\frac{1}{\pi} \frac{\partial C_{\mu_j}(z)}{\partial \bar{z}}$ is considered as a distribution, see e.g. [16, Ch. 2]. Then Theorem 5 implies that in a neighborhood of any point $p \notin \Gamma_2$ the support of μ_j coincides with a smooth 0-level curve of some function $\Re \left[\int_p^z A_i(w) dw - \int_p^z A_j(w) dw \right]$, $i \neq j$, which proves parts A and B of Theorem 3. Part C follows easily from [10, Theorem 2].

We turn to proving Theorem 4, i.e., to showing that generically, the set Γ_2 is finite. The plan is to describe the locus L of curves such that Γ_2 is finite. Now L is a semi-algebraic set, that is, L is given by real polynomial inequalities and equalities. We cannot describe these explicitly since even simple cases become quite involved, as shown by Example 3 below. Instead we will prove that L contains an open subset (in the Zariski hence also in the Euclidean topology) essentially by analyzing the construction of $\Gamma_{i,j,k}$ in real-algebraic geometry terms, as opposed to semi-algebraic. It will then suffice to find just one example of a curve in L in order to show that L is non-empty and therefore also dense, which we do in Example 3.

Clearly, $\Gamma_{i,j,k}$ is either a real analytic curve or else there exists $c \in \mathbb{R}$ such that $A_i(z) - A_k(z) = c(A_j(z) - A_k(z))$ for all $z \in U$. This latter case is certainly non-generic, and we assume that all the $\Gamma_{i,j,k}$ are real analytic curves.

First define the following variety Y , which parametrizes roots of equations of degree n . Set $P = \sum_{i=0}^k Q_i y^i \in \mathbb{C}[Q_1, \dots, Q_k, y]$ and let $Y \subset \mathbb{C}^{k+1}$ be defined by $P = 0$ and $Q_k \neq 0$. Denote by $p_1 : \mathbb{C}^{k+1} \rightarrow \mathbb{C}^k$ the projection on the k first coordinates $p_1 : (Q_1, Q_2, \dots, Q_k, y) \mapsto (Q_1, Q_2, \dots, Q_k)$ and by p_2 the projection on the last coordinate. The image of Y under p_1 is the open set $X \subset \mathbb{C}^k$ where $Q_k \neq 0$. The reason for imposing this last condition is to guarantee that $p_1 : Y \rightarrow X$ is a finite map of affine varieties.

We can also parametrize triples of roots (so we tacitly assume that $n \geq 3$) by means of the variety Y . Indeed, use p_1 to form the fiber product $Y \times_X Y \times_X Y \subset (\mathbb{C}^k \setminus V(Q_k)) \times \mathbb{C}^3$ and let

$$V := \{(y_1, y_2, y_3) \in Y \times_X Y \times_X Y \mid y_i \neq y_j \text{ if } i \neq j\}.$$

The Zariski closure \bar{V} is an open and closed subset of $Y \times_X Y \times_X Y$. The permutation group S_3 acts on the last three coordinates of $\bar{V} \subset (\mathbb{C}^k \setminus V(Q_k)) \times \mathbb{C}^3$. The orbit space $Z := \bar{V}/S_3$ can be embedded as a closed subset of $(\mathbb{C}^k \setminus V(Q_k)) \times \mathbb{C}^3$ since $\mathbb{C}^3/S_3 \cong \mathbb{C}^3$. Hence Z is a quasi-affine variety. There is a natural projection from Z to X induced by $p_1 : Y \rightarrow X$, which we denote by

$$p : Z \subset (Y \times_X Y \times_X Y)/S_3 \rightarrow X.$$

This map is finite. The fibre of p over a point $(Q_1, \dots, Q_k) \in X$ consists of the orbits of the finite set $\{A_{i,\alpha} \times A_{j,\alpha} \times A_{l,\alpha} \mid 1 \leq i, j, l \leq k\}$, where $A_{i,\alpha}$, $1 \leq i \leq k$, are the distinct solutions to the equation $P(Q, y) = 0$ under permutation of coordinates.

Any affine subvariety A of $\mathbb{C}^n = \mathbb{R}^{2n}$ can be considered as the set of \mathbb{R} -rational points of a real algebraic variety (cf. [8, Ch. 3.1]) by taking the real and imaginary parts of the complex coordinates as new coordinates. For example, \mathbb{C} itself with the ring of functions $\mathbb{C}[z]$ is the set of \mathbb{R} -rational points of the real-algebraic variety \mathbb{R}^2 with ring of functions $\mathbb{R}[x, y]$ and the correspondence is given by $z = x + iy$. For

a closed subvariety A of \mathbb{C}^n the corresponding ideal of $\mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_n]$ is

$$I_R(A) := \langle \Re P(x + iy), \Im P(x + iy) \mid P \in \mathbb{C}[z], P(z) = 0 \text{ if } z \in A \rangle.$$

Denote by $O_R(A) = \mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_n]/I_R(A)$ the ring of real-algebraic functions on A . An open subset of \mathbb{C}^n defined by the non-vanishing of a polynomial f may be considered as the set of \mathbb{R} -rational points of the real-algebraic variety defined by $\mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_n]_{\tilde{f}}$, where $\tilde{f} = (\Re f)^2 + (\Im f)^2$. More generally, any quasi-affine subvariety in \mathbb{C}^n can be thought of as the set of \mathbb{R} -rational points of a quasi-affine real-algebraic variety. We stress this since we will need the fact that the property that an algebraic map of complex quasi-affine varieties $A \rightarrow B$ is finite is preserved when turning A and B into real algebraic varieties (in the sense of algebraic geometry, meaning that $O_R(A)$ is a finite module over $O_R(B)$).

We now return to the global description of the curves $\Gamma_{i,j,k}$. By the above we can consider \bar{V} and its orbit space Z as real algebraic varieties, and as such Z contains the real algebraic subset $\bar{\Gamma}$ defined by

$$\begin{aligned} \bar{\Gamma} = \{ (y_1, y_2, y_3) \in Y \times_X Y \times_X Y \mid y_i = (\alpha, z, \gamma_i) \text{ and} \\ \Im(p_2(y_1) - p_2(y_2)) \overline{(p_2(y_3) - p_2(y_2))} = 0 \}. \end{aligned}$$

Notation 4. Consider the algebraic curve in \mathbb{C}^2 given by

$$P(z, y) = \sum_{i=0}^k Q_i(z) y^i = 0, \quad (6.1)$$

where $\deg_z Q_i \leq i$ for $i < k$ and $\deg_z Q_k = k$. Let $Q_i(z) = \sum_{0 \leq j \leq i} \alpha_{ij} z^j$, $0 \leq i \leq k$, set $\alpha = (\alpha_{ij})$ and denote the curve (6.1) corresponding to $\alpha = (\alpha_{ij})$ by L_α .

Without loss of generality we may assume that $\alpha_{k,k} = 1$. Then polynomials $P(z, y)$ are parametrized by the r -tuple $\alpha = (\alpha_{i,j}) \in \mathbb{C}^r := \mathbb{C}^k \times \mathbb{C}^k \times \mathbb{C}^{k-1} \times \dots \times \mathbb{C}$. In addition we will assume that the constant polynomial $Q_0 = \alpha_{0,0} \neq 0$. In particular, the set $V(Q_k, Q_{k-1}, \dots, Q_0) \subset \mathbb{C}$ where all the Q_i 's vanish is empty. Let $c_\alpha : \mathbb{C} \rightarrow \mathbb{C}^k$ be the map $c_\alpha(z) = (Q_1(z), \dots, Q_k(z))$. Then the pullback by c_α of $p_1 : \mathbb{C}^{k+1} \rightarrow \mathbb{C}^k$ is the curve $L_\alpha \subset \mathbb{C}^2 \mapsto \mathbb{C} \setminus V(Q_k)$, where $V(Q_k)$ is the zero set ("support") of Q_k . Hence over a point $z \in \mathbb{C}$ the pullback of $Y \times_X Y \times_X Y \rightarrow X$ will locally have the finite fibre

$$\{(z, A_i(z) \times A_j(z) \times A_l(z)) \mid 1 \leq i, j, l \leq k\},$$

where A_i , $1 \leq i \leq k$, are the distinct branches of the algebraic curve L_α .

Define the real-algebraic variety Γ as the image of $\bar{\Gamma}$ under the finite map of real-algebraic varieties $\bar{V} \rightarrow Z$.

Lemma 15. *In the above notation the following holds:*

- (i) $\bar{\Gamma}$ is preserved by the action of S_3 .
- (ii) Let $\alpha = (\alpha_{i,j})$ and

$$c_\alpha : \mathbb{C} \rightarrow \mathbb{C}^{r+1}$$

be the map such that $c_\alpha(z) = (Q_1(z), \dots, Q_k(z), z)$. Then the pullback $c_\alpha^{-1}(p(\Gamma)) \subset \mathbb{C}$ is locally the same as $\bigcup \Gamma_{i,j,k}$, where the union is taken over all indices i, j, l for the curve L_α .

Proof. The identity $\Im(u - w) \overline{(v - w)} = \Im(u\bar{v} - u\bar{w} - w\bar{v})$ gives the first part. Now $\Im(u - w) \overline{(v - w)} = 0$ if and only if $\frac{u-w}{v-w} \in \mathbb{R}$, which together with the definitions yields statement (ii) in the lemma. \square

Let R_Γ be the \mathbb{R} -algebra whose subset of \mathbb{R} -rational points coincides with Γ and let R_X be the \mathbb{R} -algebra that corresponds to X in similar fashion. Then the map p corresponds to the map $p^* : R_X \rightarrow R_\Gamma$ and the kernel J of p^* defines the image of p . Furthermore, since p^* is finite the R_X -annihilator $J_2 \supset J$ of $R_\Gamma/p^*(R_X)$, that is,

$$J_2 = \{r \in R_X \mid p(r)s = 0, s \in R_\Gamma\}$$

exists and defines the *support* $V(J_2)$ of $R_\Gamma/p^*(R_X)$. It also defines the locus of prime ideals m in $\text{Spec } R_X$ such that the fibre $R_\Gamma/p^*(m)$ is not equal to R_X/m . Thus $V(J_2)$ defines the locus where the fibre has multiplicity (strictly) greater than one. (The existence of such an ideal was the whole point of our excursion into algebraic geometry.) Note that the fact that the multiplicity of the fibre over a \mathbb{R} -rational point in X is greater than one does not guarantee the existence of \mathbb{R} -rational points in the fibre.

Now we can finally reduce the proof of Theorem 4 to finding an explicit example. Consider the map $c : \mathbb{C}^r \times \mathbb{C} \rightarrow \mathbb{C}^k$ defined by $c(\alpha, z) = (Q_1(z), \dots, Q_k(z))$. (The connection between α and Q_i is as above.) Let $A = \mathbb{C}^r \times \mathbb{C} \setminus V(Q_k)$. Take the map $c^* : R_X \rightarrow R_A$ corresponding to c . The ideal $c^*(I_2)R_A$ describes the points in $\text{Spec } R_A$ such that the fibre of the pullback has multiplicity greater than one. Denote the real coordinates in $\mathbb{C}^r \times \mathbb{C}$ by x_{st}, y_{st}, x, y such that $\alpha_{st} = x_{st} + iy_{st}$ and $z = x + iy$. A point $\alpha = (\alpha_{st}) \in \mathbb{C}^r$ corresponds to a maximal ideal m_α . By Lemma 15 the ring

$$S_\alpha := \mathbb{R}[x, y]/m_\alpha c^*(I_2)R_A$$

describes the set of points in \mathbb{C} that lie on two distinct $\Gamma_{i,j,l}$'s.

The following lemma (whose simple proof we included for completeness) proves that the property of α that S_α is finite as a \mathbb{R} -vector space is generic.

Lemma 16. *Suppose that $c : T \rightarrow S$ is a map of finitely generated \mathbb{R} -algebras and let U be the set of \mathbb{R} -rational points m in $\text{Spec } T$ such that $S/c(m)$ is a finite \mathbb{R} -algebra. Then U is open.*

Proof. Let V be a finite-dimensional vector space containing $1 \in S$ that generates S , and let V^n denote the set of sums of products of at most n elements from V with coefficients in $c(T)$. The fact that V generates S implies that $S = \bigcup_{n \geq 0} V^n$. Now $S/c(m)$ is a finite \mathbb{R} -algebra if and only if there exists a non-negative integer n such that

$$T/m \otimes_T (V^{n+1}/V^n) = 0.$$

Let $J_k := \text{Ann}_T V^{k+1}/V^k$, $k \geq 0$. Since T is a ring one has that $J_k \subset J_{k+1}$. Set $J = \bigcup_{k \geq 0} J_k$. Then U consists of all \mathbb{R} -rational points m in $\text{Spec } T$ which are not contained in the support $V(J)$. Since the latter is obviously closed this proves the lemma. \square

Applied to our situation this means that there exists a set of algebraic equations in the variables x_{st}, y_{st} which when violated will guarantee that the number of points in \mathbb{C} that lie on two distinct $\Gamma_{i,j,l}$'s is finite. Thus to conclude the proof of Theorem 4 it only remains to find an example, which we do next.

Example 3. *Let b_1, \dots, b_k be complex numbers. Consider the reducible plane curve L given by*

$$\prod_{i=1}^k ((b_i - z)y - 1) = 0.$$

This equation has the solutions $y = \frac{1}{b_i - z}$, $i = 1, \dots, k$, and it satisfies the conditions of (6.1). To describe the sets $\Gamma_{i,j,k}$ we use the following notation. Let α_i, β_i , $i = 1, 2, 3$, be our variables and define an action of the symmetric group S_3 on

polynomial functions in these variables by permuting the pairs (α_i, β_i) . If M is a monomial let η_M denote the alternating monomial function

$$\eta_M = \sum_{\sigma \in S_3} (-1)^{\text{sign}(\sigma)} \sigma M.$$

(For example, $\eta_{\alpha_1\beta_2} = \alpha_1\beta_2 - \alpha_2\beta_1 - \alpha_1\beta_3 + \alpha_2\beta_3 + \alpha_3\beta_1 - \alpha_3\beta_2$.) The set Γ for the curve L is reducible and consists of one component for each $\{i, j, k\}$. Simple calculations show that the set $\Gamma_{1,2,3}$ is a circle in \mathbb{C} consisting of all points $z = u + iv$ satisfying the equation

$$\begin{aligned} p_{1,2,3} = & -\eta_{\alpha_1\beta_2}(u^2 + v^2) + (\eta_{\alpha_1\beta_2^2} + \eta_{\alpha_1\alpha_2^2})v \\ & + (\eta_{\beta_1^2\beta_2} + \eta_{\alpha_1^2\beta_2})u + (\eta_{\alpha_1\beta_2\beta_3^2} + \eta_{\alpha_1\alpha_3^2\beta_2}) = 0. \end{aligned}$$

The center of this circle is given by

$$-\left(\frac{\eta_{\beta_1^2\beta_2} + \eta_{\alpha_1^2\beta_2}}{2\eta_{\alpha_1\beta_2}}, \frac{\eta_{\alpha_1\beta_2^2} + \eta_{\alpha_1\alpha_2^2}}{2\eta_{\alpha_1\beta_2}} \right),$$

where it is understood that $b_j = \alpha_j + i\beta_j$, $j = 1, 2, 3$, and that $\Gamma_{1,2,3}$ is non-empty (which is generically true). Hence for a generic set of complex numbers b_1, \dots, b_k all sets $\Gamma_{i,j,k}$ are circles with different centers, so that in particular the intersection of two sets $\Gamma_{i,j,k}$ corresponding to two different index sets $\{i, j, k\}$ is finite. S_α is the direct sum of the algebras $\mathbb{R}[u, v]/I$, where I is the ideal generated by two different quadrics $p_{i,j,k}$ and $p_{r,s,t}$, and it is therefore finite. This shows that the set U in Lemma 16 is non-empty.

The proof of Theorem 4 is now complete.

7. FINAL REMARKS AND PROBLEMS

The main topic of the paper is intimately related to the classical asymptotic theory of linear ordinary differential equations and with the WKB-method. These are covered in a huge bulk of both older and modern literature including the classical books by J. Ecalle [13], A. Erdélyi [14], M. Fedoryuk [15], B. Malgrange [22], F. Olver [24], Y. Sibuya [26], W. Wasow [29]–[30] as well as important contributions by T. Aoki, Y. Takei, T. Kawai, M. Berry, E. Delabaere, F. Pham, Y. Colin de Verdière, A. Voros and many others, see e.g. [6], [7], [12], [25], [28]. Polynomial solutions to linear differential equations are also of special interest in connection with the so-called quasi-exactly solvable models in quantum mechanics, see papers by A. Turbiner, e.g. [27] and references therein. They are also the classical object of study in the theory of orthogonal polynomials which recently acquired new effective tools to investigate the asymptotical properties of families of such solutions, see [11].

The starting point of our study was a number of rather surprising numerical experiments in calculating the zeros of polynomial eigenfunctions for the usual spectral problem which led to several conjectures presented in [23]. Most of these conjectures were later proved in [4]. The idea to consider the homogenized spectral problem comes mainly from [15], [24] and references [1], [2].

By carefully going through the proofs of the above theorems one can see that actually these apply to a wider class of *ES*-pencils. Namely, consider *ES*-pencils of the form $T_\lambda = \sum_{i=0}^k Q_i(z, \lambda) \frac{d^i}{dz^i}$, where $Q_i(z, \lambda)$ are polynomials in z and λ . Assume that $\deg_\lambda Q_i(z, \lambda) \leq k - i$ and that $\deg_z Q_i(z, \lambda) \leq i$. Decompose $Q_i(z, \lambda) = Q_{i,0}(z) + \dots + Q_{i,k-i}(z) \lambda^{k-i}$. If the homogenized pencil $\tilde{T}_\lambda = \sum_{i=0}^k Q_{i,k-i}(z) \lambda^{k-i} \frac{d^i}{dz^i}$ is of general type then all the results in this paper apply. However, for the sake of brevity we have chosen to focus only on the homogeneous case.

Let us briefly formulate a number of related problems.

1. Stokes lines. One of the major objects in the classical asymptotic theory is the *global Stokes line* of a given linear differential equation, see e.g. [1], [2]. The theory of *global Stokes lines* is well understood in the physically important case of second order linear odes. By contrast with the latter case the development of this theory for higher order equations experiences serious difficulties. These are mostly due to the appearance of new Stokes lines (discovered in [5]) originating from the intersection points of the initial Stokes lines which emanate from the turning points of the considered equation. The supports of the asymptotic measures μ_j studied in the present paper show many similar features to the global Stokes line of a linear differential operator (1.1) which leads to the following conjecture.

Conjecture 2. *The union of the supports of the measures μ_1, \dots, μ_k is a subset of the global Stokes line for the differential operator (1.1).*

2. Convergence and topology of supports. The main open questions related to the results of the present paper are Conjecture 1 (see the introduction) and a complete description of the topological properties of the support of the limiting measures μ_j , $j = 1, \dots, k$. (We strongly believe that μ_j is unique for each j .) In particular, the following problem appears to be of fundamental importance and all the numerical evidence we obtained so far is in favor of this conjecture.

Conjecture 3. *For each j the set $\mathbb{C} \setminus \text{supp}(\mu_j)$ is connected.*

Problem 3. *Is it true that the support of each μ_j is topologically a tree?*

In general, it would be highly desirable to describe the stratification of the space ES_k according to the topology of the supports of the k -tuple of measures μ_1, \dots, μ_k .

3. Other classes of operators. A further natural question is as follows.

Problem 4. *Which of the above results hold for operators (1.1) of nongeneral type?*

Numerical experiments carried out in [3] show the existence of asymptotic root-counting measures even for such operators. However, these measures usually have noncompact supports, which leads to considerable additional complications since the arguments in all our proofs do not extend *mutatis mutandis* to such situations.

4. Positive Cauchy transforms. As we already mentioned in the introduction, Problem 2 is completely solved only for rational functions.

Problem 5. *Characterize algebraic germs of real (respectively, positive) Cauchy transforms.*

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